73. *Nonlinear Perron-Frobenius Problem for Order-preserving Mappings. II. —Applications*

By Toshiko OGIWARA

Department of Mathematical Sciences, University of Tokyo

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Abstract: In this paper, we apply the results in part I of this paper to boundary value problems for a class of partial differential equations. First, we generalize the Fujita lemma, which is concerned with the properties of solutions of the equation $\Delta u + f(u) = 0$ with strictly convex function f, to the case where f is a convex function. The second example is a bifurcation problem for the semilinear elliptic equation of the form $\Delta u + \lambda f(u) = 0$ under the Dirichlet boundary conditions. We discuss properties of a bifurcation branch of solutions. The third example is a nonlinear (but positively homogeneous) eigenvalue problem.

Key words: Perron-Frobenius; order-preserving; indecomposable; bifurcation; generalized Fujita lemma.

1. Introduction. In part I of the present series of papers, we have extended the Perron-Frobenius theorem to nonlinear mappings on an infinite dimensional space. We have studied the properties of eigenvalues and the corresponding eigenvectors.

In this paper we apply the results in part I to a class of partial differential equations.

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2. Generalized Fujita lemma. In what follows, the numbered 'theorems' and 'remarks', as well as the hypotheses A1, A2, A3, $\cdots$, refer to those presented in part I.

Example 1. Let $\Omega \subset \boldsymbol{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the Dirichlet boundary value problem:

$$\begin{cases} \Delta u + f(x, u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where φ is a continuous function on $\partial\Omega$. Here $f(x, u) : \bar{\Omega} \times \boldsymbol{R} \to \boldsymbol{R}$ is locally Hölder continuous in x, u, and locally uniformly Lipschitz continuous in u, that is, for any bounded closed interval $[a, b] \subset \boldsymbol{R}$, there exists some constant $C > 0$ such that

$$\sup_{x \in \Omega} \sup_{\substack{u,v \in [a,b] \\ u \neq v}} \frac{|f(x, u) - f(x, v)|}{|u - v|} \leq C.$$

Hereafter we consider only classical solutions. (It is easily shown that any bounded weak solution is classical.) For two solutions u and v, we write $u \leq v$ if $v(x) - u(x) \geq 0 (x \in \bar{\Omega})$, $u < v$ if $u \leq v$ and $u \neq v$, and $u \ll v$ if $v(x) - u(x) > 0 (x \in \Omega)$ and $\partial v / \partial n(x) - \partial u / \partial n(x) < 0 (x \in \partial\Omega)$. Here $\partial / \partial n(x)$ denotes the outer normal derivative at $x \in \partial\Omega$.

We have the following:

Proposition 1. *Suppose that $u \mapsto f(x, u)$ is concave for each $x \in \Omega$. Let u_1, u_2, u_3 be solutions of* (1.1) *satisfying $u_1 < u_2$ and $u_1 < u_3$. Then either* (i), (ii) *or* (iii) *holds.*

(i) $u_1 \ll u_2 = u_3$

(ii) $u_1 \ll u_2 \ll u_3$ *and* $u_2 = \bar{r} u_1 + (1 - \bar{r}) u_3$ *for some* $\bar{r} \in (0, 1)$. *Furthermore, for any* $r \in [0, 1]$,

$$f(x, ru_1(x) + (1 - r)u_3(x)) = rf(x, u_1(x)) + (1 - r)f(x, u_3(x))$$

(*hence* $ru_1(x) + (1 - r)u_3(x)$ *is a solution of* (1.1)).

(iii) $u_1 \ll u_3 \ll u_2$ *and* $u_3 = \bar{r} u_1 + (1 - \bar{r}) u_2$ *for some* $\bar{r} \in (0, 1)$. *The same statement as* (ii) *holds with* u_2, u_3 *exchanged each other.*

Remark. The statement of the proposition remains true with the relations $\leq$, $<$ and $\ll$ replaced by $\geq$, $>$ and $\gg$, respectively, if the concavity assumption on $u \mapsto f(x, u)$ is replaced by the convexity assumption. To see this, simply replace u by $-u$, $f(x, u)$ by $-f(x, -u)$.

Outline of the proof. Put $g(x, w) = f(x, w + u_1(x)) - f(x, u_1(x))$ and let us consider the following problem:

$$\begin{cases} \Delta w + g(x, w) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Obviously u is a solution of (1.1) if and only if $u - u_1$ is a solution of (1.2).

Since u_1, u_2 and u_3 are continuous functions, there exists some constant $k > 0$ such that $g(x, w) + kw$ is strictly increasing in $w \in [0, \bar{w}(x)]$ for each $x \in \bar{\Omega}$. Here $\bar{w}(x) = \max\{u_2(x) - u_1(x), u_3(x) - u_1(x)\}$. Clearly the function $\bar{w} \neq 0$.

Let $E = L^p(\Omega)$, $E_+ = L^p(\Omega)_+ = \{u \in L^p(\Omega) \mid u(x) \geq 0 \text{ a.e. } x \in \Omega\}$ and $V = C^1(\bar{\Omega}) \cap C_0(\bar{\Omega})$. Here $C_0(\bar{\Omega})$ denotes the space of continuous functions on $\bar{\Omega}$ vanishing on the boundary $\partial\Omega$. Note that the positive cones in the space $L^p(\Omega)$ or $C_0(\bar{\Omega})$ have empty interior, whereas $C^1(\bar{\Omega}) \cap C_0(\bar{\Omega})$ has a positive cone with nonempty interior. On the other hand, the norm in $L^p(\Omega)$ or $C_0(\bar{\Omega})$ has the monotonicity as defined in (4.1) of part I, while that of $C^1(\bar{\Omega}) \cap C_0(\bar{\Omega})$ does not have such a property.

Set

$$\bar{g}(x, w) = \begin{cases} g(x, w) + kw & \text{if } w \leq \bar{w}(x) \\ g(x, \bar{w}(x)) + k\bar{w}(x) & \text{if } w > \bar{w}(x) \end{cases}$$

and define the mapping $T : E_+ \to V_+$ by

$$Tw = (-\Delta_D + k)^{-1} \bar{g}(x, w(x)).$$

Here Δ_D denotes the Laplace operator under the Dirichlet boundary conditions. Taking $p > n$ and using the Sobolev embedding theorem and the maximum principle, we find that $T : E_+ \to V_+$ satisfies the hypotheses B1, A2′, A3 and the property defined in Remark 9 in part I. Note that w is a solution of (1.2) satisfying $0 \leq w \leq \bar{w}$ if and only if w is an eigenvector of T corresponding to 1. Applying the generalized version of Theorems 5, 6 and Remark 9 to this case, we obtain the conclusion of the proposition.

The following is an immediate consequence of Proposition 1 and the subsequent remark.

Corollary (a generalized Fujita lemma). *Suppose that* $u \mapsto f(x, u)$ *is concave for each* $x \in \Omega$ *or convex for each* $x \in \Omega$. *Let* u_1, u_2, u_3 *be solution of* (1.1) *satisfying* $u_1 \leq u_2 \leq u_3$. *Then either* (a), (b) *or* (c) *holds.*

(a) $u_1 = u_2 = u_3$.

(b) $u_1 = u_2 \ll u_3$ *or* $u_1 \ll u_2 = u_3$.

(c) $u_1 \ll u_2 \ll u_3$ *and statement* (ii) *of Proposition* 1 *holds.*

3. Bifurcation problem. Example 2. Next we consider the following problem:

$$
\begin{cases} \Delta u + \lambda f(v) = 0 & \text{in } \Omega, \\ \Delta v + \lambda g(u) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}
$$

Here $f : \boldsymbol{R} \to \boldsymbol{R}$ is a locally Hölder continuous function satisfying

(F.1) $f(0) = 0$, $f_* := \lim_{\rho \searrow 0} \{f(\rho)/\rho\} > 0$,

(F.2) $0 \leq f(\alpha u) \leq \alpha f(u)$ for any $\alpha > 1$, $u > 0$,

(F.3) $f(u)$ is nondecreasing in $u > 0$.

We also assume that $g : \boldsymbol{R} \to \boldsymbol{R}$ satisfies precisely the same conditions as above and denote those conditions by (G.1), (G.2), (G.3).

Let $\tilde{E} = E \times E$, $\tilde{E}_+ = E_+ \times E_+$ and $\tilde{V} = V \times V$ with E, E_+, V defined in Example 1.

In what follows we consider the number λ in (2.1) to be an unspecified constant, therefore each solution of (2.1) will be written in the form of a pair $(\lambda, (u, v))$. Obviously $(\lambda, (0, 0))$ is a solution of (2.1) for all $\lambda \in \boldsymbol{R}$, which we call a "trivial solution". We say $(\lambda, (u, v))$ is a "positive solution" if $(u, v) > 0$.

The pair $(\lambda, (u, v))$ is a positive solution of (2.1) if and only if $(u, v) > 0$, and satisfies

$$\lambda T(u, v) = (u, v),$$

where T is defined by

$$T(u, v) = ((-\Delta_D)^{-1} f(v), (-\Delta_D)^{-1} g(u)).$$

By the same way as that of Example 1, we find that $T : \tilde{E}_+ \to \tilde{V}_+$ satisfies the hypotheses B1, A2′, A3 and the property defined in Remark 9. Applying the generalized version of Theorem 6 and Remarks 8, 9 to this case, we obtain the following:

Proposition 2. (i) *Positive solutions of* (2.1) *bifurcate at* $(\lambda_1^*, (0, 0))$ *from the trivial solutions. Here* $\lambda_1^* = \lambda_1 / \sqrt{f_* g_*}$, *where* λ_1 *is the smallest eigenvalue of* $-\Delta_D$. *There is no other bifurcation point. Moreover, there exist mappings* $\lambda : (0, \infty) \to \boldsymbol{R}_+$ *and* $(u, v) : (0, \infty) \to \tilde{E}_+$ *such that* $\{(\lambda(\rho), (u(\rho), v(\rho))) \mid \rho \in (0, \infty)\}$ *coincides with the set of all positive solutions of* (2.1). *Furthermore,* λ *is a nondecreasing, subhomogeneous and continuous function, while* (u, v) *is continuous and satisfies* $\| u(\rho) \|_{L^p} + \| v(\rho) \|_{L^p} = \rho$.

(ii) *In addition to condition* (F.2), *assume further that f satisfies*

(F.2$'$) there exists some $\delta > 0$ such that $0 \leq f(\alpha u) < \alpha f(u)$ for any $\alpha > 1$, $u \in (0, \delta)$,

or that g satisfies the same condition as above in addition to (G.2). *Then $\lambda(\rho)$ is strictly increasing in ρ and hence (u, v) is parametrizable by λ.*

Remark. Proposition 2 deals with properties of a bifurcation branch of positive solutions of (2.1). Our method, of course, is also applicable to the single equation

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

which can be handled more easily than the system (2.1). The results for these problems are to some extent known, particularly those for (2.2). But our proof has an advantage in that it requires weaker regularity, monotonicity and subhomogeneity assumptions than those results found in the literature (such as [6], [7]).

4. Nonlinear eigenvalue problem. Example 3.a. We consider the following problem:

$$\begin{cases} \Delta u + \lambda(u \vee v + \varepsilon v) = 0 & \text{in } \Omega, \\ \Delta v + \lambda(a(u \wedge v) + \varepsilon b u) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $a, b, \varepsilon > 0$ are constants. Here we use the notation

$(u \vee v)(x) = \max\{u(x), v(x)\}$ and $(u \wedge v)(x) = \min\{u(x), v(x)\}$.

The pair $(\lambda, (u, v))$ is a positive solution of (3.1) if and only if $(u, v) > 0$, and satisfies

$$\lambda T(u, v) = (u, v),$$

where T is defined by

$$T(u, v) = ((-\Delta_D)^{-1}(u \vee v + \varepsilon v), (-\Delta_D)^{-1}(a(u \wedge v) + \varepsilon b u)).$$

The same argument as that of Example 2 shows that $T : \tilde{E}_+ \to \tilde{V}_+$ satisfies the hypotheses B1, A2, A3 and B4. Applying the generalized version of Theorem 4 to this case, we obtain the following:

Proposition 3. *The equation* (3.1) *has a positive solution $(\lambda, (u, v))$ such that $\lambda = 1/r(T) > 0$. Furthermore, (u, v) is unique up to multiplication by a positive constant. There is no other positive solution.*

Example 3.b. It is interesting to study what happens when we let ε tend to 0 in (3.1). The limit equation does no longer satisfy condition B4, therefore the existence of a positive eigenfunction does not follow from our general theory. In order to answer the above question in a more general framework, let us consider the following problem:

$$\begin{cases} \Delta u + \lambda(S_1(u) \vee S_2(v)) = 0 & \text{in } \Omega, \\ \Delta v + \lambda(S_3(u) \wedge S_4(v)) = 0 & \text{in } \Omega, \\ u = v = 0 & \text{in } \partial\Omega. \end{cases} \tag{3.2}$$

Here $S_i(u)(x) = a_i(x)u(x)$, where $a_i : \bar{\Omega} \to \boldsymbol{R}$ is a continuous function satisfying $a_i(x) > 0$ for any $x \in \bar{\Omega} (i = 1, 2, 3, 4)$.

Define $T : \tilde{E}_+ \to \tilde{V}_+$ by

$$T(u, v) = ((-\Delta_D)^{-1}(S_1(u) \vee S_2(v)), (-\Delta_D)^{-1}(S_3(u) \wedge S_4(v))),$$

and denote the composed mappings $(-\Delta_D)^{-1}\circ S_i$ by T_i for $i=1, 2, 3, 4$. Each $T_i : E_+ \to V_+$ satisfies the hypotheses B1, A2, A3 and B4, and T satisfies the hypotheses B1, A2, A3. Note, however, that T does not necessarily satisfy B4.

In what follows by a "solution" we mean a normalized solution satisfying $\| u \|_V + \| v \|_V = 1$. We have the following:

Proposition 4. *The equation* (3.2) *has a positive solution* $(1/r(T_1), (u, 0))$ *for some* $u \in (V_+)^i$, *where* $r(\cdot)$ *denotes the quantity defined in* (2.2) *of part I.*

(i) *Assume* $r(T_1) < r(T_4)$. *Then there exists some* $(\tilde{u}, \tilde{v}) \gg 0$ *such that* $(1/r(T), (\tilde{u}, \tilde{v}))$ *is a solution. Moreover, the pairs* $(1/r(T_1), (u, 0))$ *and* $(1/r(T), (\tilde{u}, \tilde{v}))$ *are the only two positive solutions.*

(ii) *Assume* $r(T_1) = r(T_4)$. *Then there exist some* $\tilde{v} \in (V_+)^i$ *and* $0 < c' < 1$ *such that* $(1/r(T_1), ((1-c)u, c\tilde{v}))$ *is a solution for any* $0 \leq c \leq c'$. *There is no other positive solution.*

(iii) *Assume* $r(T_1) > r(T_4)$. *Then* $(1/r(T_1), (u, 0))$ *is a unique positive solution.*

Remark. Put $a_1(x) = a_2(x) = 1$ and $a_3(x) = a_4(x) = a$ with constant $a > 0$. Proposition 4 shows that the normalized positive solution of Example 3.a converges to $(\tilde{u}, \tilde{v})$, or $(u, 0)$ as ε tends to 0 if $a > 1$, or $a < 1$, respectively. In the case where $a = 1$, by a simple calculation, we also find that it converges to $((1-c)u, c\tilde{v})$ with $c = c' = 1/2$, or $c = \sqrt{b}/(1+\sqrt{b})$ if $b \geq 1$, or $b < 1$, respectively. Here $\tilde{u}, \tilde{v}, u$ are those in Proposition 4.

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