72. Nonlinear Perron-Frobenius Problem for Order-preserving Mappings. I

By Toshiko OGIWARA

Department of Mathematical Sciences, University of Tokyo

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Abstract: We consider the eigenvalue problem of an order-preserving mapping defined on a positive cone of an ordered Banach space. Among other things, we prove the existence and, in some cases, the uniqueness of the positive eigenvalue. We also discuss other properties of eigenvalues and eigenvectors. The notion of indecomposability for nonlinear mappings that we introduce in an infinite dimensional setting will play a key role in our argument. We apply the results in this paper to boundary value problems for a class of partial differential equations in part II.

Key words: Perron-Frobenius; order-preserving; indecomposable; maximal eigenvalue; positive eigenvector.

1. Introduction. The Perron-Frobenius theorem, which is concerned with the properties of eigenvalues and eigenvectors of square matrices whose components are nonnegative, has been extended and applied in various ways. It has been generalized to positive linear operators on a Banach space in [1], [3], [6], [12]. From the point of view of applications to mathematical economics, extensions of the theory to nonlinear mappings have also been obtained in [4], [5], [9], [10], [11]. They are, however, concerned only with problems in a finite dimensional Euclidean space.

In this paper, we extend these results to nonlinear mappings on an infinite dimensional space. In doing so, we introduce the notion of indecomposability for a nonlinear mapping on an infinite dimensional space. This notion is an infinite dimensional extension of that for a mapping on an n-dimensional Euclidean space defined in [4; Appendix], and is also a nonlinear extension of that for a linear operator found in [2]. We then consider the eigenvalue problem of an order-preserving mapping T defined on a positive cone E_+ of an ordered Banach space E. We define the operator norm of a positively homogeneous mapping T and denote $\lim_{n\to\infty} \| T^n \|^{1/n}$ by $r(T)$, as in the case of linear operator. The quantity $r(T)$ plays an important role in establishing the existence of positive eigenvalues. This places our problem in marked contrast with the case of finite dimensional spaces, in which the estimation of $r(T)$ is of little importance since the existence of positive eigenvalues is obtained by rather a straightforward application of Brower fixed-point theorem.

As the space is limited, we omit the proof of our theorems. See the forthcoming paper [7] for details.

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2. Notations and assumptions. Let E be an ordered Banach space, that is, a real Banach space provided with an *order cone* E_+ (a closed convex cone with vertex at 0 such that $E_+ \cap (-E_+) = \{0\}$). We assume that the interior of E_+, denoted by $(E_+)^i$, is nonempty. Such a space is called a *strongly ordered* Banach space. We also assume that $\dim E \geq 2$.

For $x, y \in E$ we write $x \gg y$ if $x - y \in (E_+)^i$, $x > y$ if $x - y \in E_+ \setminus \{0\}$, and $x \geq y$ if $x - y \in E_+$. For $x \in E$ we say that x is *strongly positive, positive, nonnegative* if and only if $x \gg 0$, $x > 0$, $x \geq 0$, respectively.

We assume that the norm on E is monotone, namely,

(2.1) $$0 \leq x \leq y \text{ implies } \|x\| \leq \|y\|.$$

For $x \geq 0$, we denote

$$E_x = \{y \geq 0 \mid y \leq \lambda x \text{ for some } \lambda > 0\}.$$

Note that $E_x = \{0\}$ if and only if $x = 0$, and $E_x = E_+$ if and only if $x \gg 0$.

Let T be a mapping from E_+ into itself. We will impose on T the following conditions:

A1(compactness): T is continuous and the image of a bounded set by T is relatively compact,

A2(positive homogeneity): $T(\lambda x) = \lambda Tx$ for any $\lambda > 0$, $x \geq 0$,

A2′(subhomogeneity): $T(\lambda x) \leq \lambda Tx$ for any $\lambda > 1$, $x \geq 0$,

A3(order-preserving property): $x \leq y$ implies $Tx \leq Ty$,

A4(indecomposability): $\{0\} \subsetneq E_{x-y} \subsetneq E_+$ implies $Tx - Ty \notin E_{x-y}$.

It is often useful to express the indecomposability condition in the following form:

Lemma 1. *Assume* A4 *and let* $x \geq y$. *Then there exists a constant* $\lambda > 0$ *such that*

$$Tx - Ty \leq \lambda(x - y)$$

if and only if either $x = y$ *or* $x \gg y$.

We define

$$VP(T) = \{\lambda \mid \lambda \text{ is an eigenvalue of } T\} = \{\lambda \mid Tx = \lambda x \text{ for some } x > 0\}$$

and denote the set of eigenvectors corresponding to λ by W_λ. We then set

$$W = \bigcup_{\lambda \in VP(T)} W_\lambda.$$

For each $\rho > 0$ we denote $S_\rho = \{x \geq 0 \mid \|x\| = \rho\}$ and

$$\lambda_\rho(T) = \begin{cases} \sup VP(T|_{S_\rho}) & \text{if } VP(T|_{S_\rho}) \neq \emptyset, \\ -\infty & \text{if } VP(T|_{S_\rho}) = \emptyset, \end{cases}$$

where $T|_{S_\rho}$ means the restriction of T on S_ρ. We also set

$$W(\rho) = W_{\lambda_\rho(T)} \cap S_\rho.$$

This defines a multivalued mapping $W : (0, \infty) \to 2^{E_+}$. It is clear that $\lambda_\rho(T)$ is independent of ρ if T is positively homogeneous. Finally we set

$$\|T\| = \sup\{\|Tx\| \mid x \geq 0 \text{ and } \|x\| = 1\}.$$

Lemma 2. *Under the assumptions* A2 *and* A3, $\|T\| < +\infty$. *Furthermore,* $\lim_{n\to\infty} \|T^n\|^{1/n}$ *exists and satisfies* $\lim_{n\to\infty} \|T^n\|^{1/n} \leq \|T\|$.

Denote

$$r(T) = \lim_{n\to\infty} \| T^n \|^{1/n} . \tag{2.2}$$

3. Main results. First we show the properties of the maximal eigenvalue of positively homogeneous mapping T and study the relation between the maximal eigenvalue and $r(T)$.

Theorem 3. *Suppose that $T : E_+ \to E_+$ satisfies the assumptions* A1, A2, A3 *and that $r(T) > 0$. Then $r(T)$ is the maximal eigenvalue of T.*

Theorem 4. *Let T satisfy the assumptions* A1, A2, A3, *and* A4. *Then $r(T) > 0$, and is the only eigenvalue of T. The corresponding eigenvector is strongly positive and unique up to multiplication by a positive constant.*

Next, we assume that the mapping $T : E_+ \to E_+$ satisfies the condition A2′ (subhomogeneity), which is a relaxed version of A2.

Theorem 5. *Let T satisfy the assumptions* A1, A2′, A3, *and have a positive eigenvalue. Then*

(i) *$\lambda_\rho(T) > 0$, and is an eigenvalue of T for each $\rho > 0$,*

(ii) *$\lambda_\rho(T)$ is continuous and nonincreasing with respect to $\rho > 0$,*

(iii) *$W(\rho)$ is nonempty and compact for each $\rho > 0$,*

(iv) *the correspondence $\rho \mapsto W(\rho)$ is upper semicontinuous in the sense of a multivalued mapping.*

Theorem 6. *Let T satisfy the assumptions* A1, A2′, A3 *and* A4. *Then T has an eigenvalue and*

$$VP(T) = \{\lambda_\rho(T) \mid \rho > 0\} \not\ni 0, \quad W = \cup_{\rho>0} W(\rho).$$

Further the following properties hold:

(i) *$W(\rho)$ is a singleton for each $\rho > 0$,*

(ii) *$0 \ll W(\rho) \ll W(\rho')$ when $0 < \rho < \rho'$,*

(iii) *$\rho \mapsto W(\rho)$ is a continuous mapping from $(0, \infty)$ to E_+.*

Finally, we study the generalized eigenvalue problem:

$$T_\lambda u = u, \tag{3.1}$$

where $\{T_\lambda\}_{\lambda>0}$ is a family of mappings with positive real parameter $\lambda > 0$.

Theorem 7. *Let each $T_\lambda : E_+ \to E_+$ satisfies* A1, A2′, A3 *and* A4, *that*

$$T_\lambda(\mu x) < \mu T_\lambda x \quad (\mu > 1,\ x \gg 0), \tag{3.2}$$

and that there exists a positive solution $u > 0$ of (3.1) *for some $\lambda > 0$. Then there exists a continuous mapping $u(\lambda)$ from some interval Λ into E_+ such that*

$$\{(\lambda, u) \mid T_\lambda u = u \text{ and } u > 0\} = \{(\lambda, u(\lambda)) \mid \lambda \in \Lambda\}.$$

The above set is unbounded in $(0, \infty) \times E_+$. Furthermore, $u(\lambda) \leq u(\lambda')$ for any $0 < \lambda < \lambda'$ —— hence, in particular, $\| u(\lambda) \|$ is nondecreasing in λ —— and, if $\lambda_* = \inf \Lambda > 0$,

$$\| u(\lambda) \| \to 0 \text{ as } \lambda \searrow \lambda_*.$$

Remark 8. If the inequality in A2′ in Theorem 6 holds strictly, that is,

$$T(\lambda x) < \lambda T_x$$

for any $\lambda > 1$ and $x \gg 0$, then $\lambda_\rho(T)$ is strictly decreasing with respect to $\rho > 0$.

Remark 9. The assumption A4 in Theorems 4, 6 and 7 can be relaxed somewhat. To be more precise, instead of assuming that T satisfies A4, assume simply that $W \neq \emptyset$ and that $T|_{W \cup \{0\}}$ satisfies A4, that is,

A4′: for $x, y \in W \cup \{0\}$, $\{0\} \subsetneq E_{x-y} \subsetneq E_+$ implies $Tx - Ty \notin E_{x-y}$.
Then the same statements as those of Theorems 4, 6 or 7 hold, respectively.

4 Eigenvalue problem for the case where the positive cone has empty interior. In this section we deal with the case where the ordered Banach space E is not necessarily a strongly ordered one; in other words, E_+ may have empty interior.

We assume that

(4.1) $$0 \leq x \leq y \text{ implies } \|x\| \leq \|y\|$$

and that there exists some strongly ordered Banach space $V(\dim V \geq 2)$, embedded continuously into E, with a positive cone $V_+ = E_+ \cap V$ such that $TE_+ \subset V_+$. We do not assume that $0 \leq x \leq y$ implies $\|x\|_V \leq \|y\|_V$ for $x, y \in V$, where $\|\cdot\|_V$ denotes the norm on V. For $x, y \in E$, we write $x \gg y$ if and only if $x, y \in V$ and $x - y \in (V_+)^i$. We say $x \in E$ is strongly positive if $x \gg 0$.

We replace some of the assumptions given in Section 2 by the following:
B1(compactness): $T : (E_+, \|\cdot\|) \to (V_+, \|\cdot\|_V)$ is a compact mapping,
B4(indecomposability): for $x, y \in V$,

$$\{0\} \subsetneq E_{x-y} \cap V \subsetneq V_+ \text{ implies } Tx - Ty \notin E_{x-y}.$$

Lemma 10. *Suppose that* T *satisfies the assumptions* B1, A2, A3. *Then* $\|T\| < +\infty$. *Furthermore* $\lim_{n\to\infty} \|T^n\|^{1/n}$ *exists and* $\lim_{n\to\infty} \|T^n\|^{1/n} \leq \|T\|$. *Thus* $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$ *is well defined.*

Replacing the assumption A1 by B1 and A4 by B4, we can prove the same statements as those of Theorems 3-7 and Remarks 8, 9.

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