# 65. *On a Conjecture on Pythagorean Numbers. II*

By Kei TAKAKUWA and You ASAEDA

Department of Mathematics, Gakushuin University

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 12, 1993)

In [1] we considered the following diophantine equation on $l, m, n \in \boldsymbol{N}$

(1) $$(4a^2 - y^2)^l + (4ay)^m = (4a^2 + y^2)^n$$

where $a, y \in \boldsymbol{N}$, with $(a, y) = 1$, $2a > y$, $y \equiv 3 \pmod 4$. $l$ is easily seen to be even. If $a$ is odd, then $m \neq 1 \Leftrightarrow n$ is even. If $a$ is even, then both $m$ and $n$ are even. (Cf. [1] Props. 1-3.) In this paper we consider the case $y = 3$.

**Theorem 1.** *Let $a$ be even, $a = 2^s a_0$, $(s \geq 1)$, $(2, a_0) = 1$. If the diophantine equation on $f$, $g$*

(2) $$4a^2 + 9 = (2^{s+1}f)^2 + (3g)^2$$

*has the unique solution $f = a_0$, $g = 1$, then $(l, m, n) = (2, 2, 2)$.*

**Remark.** All even $a$, with $(a, 3) = 1$, $2 \leq a \leq 152$, except 14, 46, 52, 62, 118, 142, 148, satisfy the above condition.

*Proof.* As $l, m, n$ are even, put $l = 2l'$, $m = 2m'$, $n = 2n'$ and $(4a^2 + 9)^{n'} + (4a^2 - 9)^{l'} = A$, $(4a^2 + 9)^{n'} - (4a^2 - 9)^{l'} = B$. Then it is proved in [1] that the possibility on choice of $A$, $B$ in

(3) $$2^{2m} 3^m a^m = AB$$

is only the following:

$$A = 2^{m(2+s)-1} b^m, \quad B = 2 \cdot 3^m c^m,$$

where $a_0 = bc$, $(b, c) = 1$, hence $l'$ is odd. (Cf. [1].) If $n'$ is even, then from $A = 2^{m(2+s)-1} b^m$, we have $8 \equiv 0 \pmod{16}$, which is a contradiction. Thus $n'$ is odd, too.

$$(A + B)/2 = (4a^2 + 9)^{n'} = (2^{m'(2+s)-1} b^{m'})^2 + (3^{m'} c^{m'})^2.$$

So

(4) $$(4a^2 + 9)^{n'} = (2^{m'(2+s)-1} b^{m'} + 3^{m'} c^{m'} i)(2^{m'(2+s)-1} b^{m'} - 3^{m'} c^{m'} i).$$

Put $F = 2^{m'(2+s)-1} b^{m'} + 3^{m'} c^{m'} i$, $G = 2^{m'(2+s)-1} b^{m'} - 3^{m'} c^{m'} i$. Then $1 = (F, G)$, as $(b, c) = (b, 6) = (c, 6) = 1$. Therefore there exist integers $f_0$, $g_0$ such that $(f_0, g_0) = 1$, $F = (f_0 + g_0 i)^{n'}$, hence $4a^2 + 9 = f_0^2 + g_0^2$. By Lemma 1, which we prove below, we have $3 \mid g_0$, $2^{m'(2+s)-1} \parallel f_0$, so $2^{s+1} \mid f_0$. By the assumption we have $f_0 = 2a$, $g_0 = 3$. Since $2^{m'(2+s)-1} \parallel 2a$, $m'(2 + s) - 1 = s + 1$. Thus $m' = 1$, so $m = 2$. Then $A = (4a^2 + 9)^{n'} + (4a^2 - 9)^{l'} = 2^{2(2+s)-1} b^2 \leq 2^{2(2+s)-1} a_0^2 = 8a^2 = (4a^2 + 9) + (4a^2 - 9)$. Therefore $n' = l' = 1$. Thus $(l, m, n) = (2, 2, 2)$.

**Lemma 1.** *Let $a$ be even and $a_0, s, b, c, m', n', F, G$ as above. If integers $f$, $g$ with $(f, g) = 1$ satisfy $4a^2 + 9 = f^2 + g^2$ and $2^{m'(2+s)-1} b^{m'} + 3^{m'} c^{m'} i = (f + gi)^{n'}$, then $2^{m'(2+s)-1} \parallel f$, $3 \mid g$.*

*Proof.*

$$(f + gi)^{n'} = \sum_{j=0}^{(n'-1)/2} \binom{n'}{2j} f^{n'-2j} (-1)^j g^{2j} + ig \sum_{j=0}^{(n'-1)/2} \binom{n'}{2j+1} f^{n'-(2j+1)} (-1)^j g^{2j}.$$

Therefore

$$\text{(i)}\quad 2^{m'(2+s)-1}b^{m'} = f\sum_{j=0}^{(n'-1)/2}\binom{n'}{2j} f^{n'-(2j+1)}(-1)^j g^{2j},$$

$$\text{(ii)}\quad 3^{m'}c^{m'} = g\sum_{j=0}^{(n'-1)/2}\binom{n'}{2j+1} f^{n'-(2j+1)}(-1)^j g^{2j}.$$

Since $f^2 + g^2 = 4a^2 + 9$ is odd, $f \not\equiv g \pmod 2$. Then $g$ is odd and $f$ is even from (ii). Therefore $\sum_{j=0}^{(n'-1)/2}\binom{n'}{2j} f^{n'-(2j+1)}(-1)^j g^{2j}$ is odd, hence we have $2^{m'(2+s)-1} \| f$ from (i).

Assume $3 \mid f$, then from (ii), $3 \mid g$, which contradicts the assumption $(f, g) = 1$. Therefore $3 \nmid f$. As $3 \nmid a$, too, $a^2 \equiv f^2 \equiv 1 \pmod 3$. Hence $g^2 = -f^2 + 4a^2 + 9 \equiv 0 \pmod 3$. Thus $3 \mid g$.

**Theorem 2.** *Let $a$ be a square free odd integer. If the class number of $k = \boldsymbol{Q}(\sqrt{-3a})$ is a power of 2, then $(l, m, n) = (2, 2, 2)$.*

**Remark.** All square free odd $a$, with $(a, 3) = 1$, $5 \leq a \leq 97$, except 29, 43, 53, 67, 77, 79, 83, 85, satisfy the class number condition. (Cf. [2].)

*Proof.* Suppose that $n$ is odd. Then $m = 1$.

Case (i) $a \equiv 3 \pmod 4$ : As $a \neq 3$ we have $a \geq 7$. Put $l = 2l'$. Clearly $n \geq 3$. From (1)

$$((4a^2 - 9)^{l'} + 2\sqrt{-3a})((4a^2 - 9)^{l'} - 2\sqrt{-3a}) = (4a^2 + 9)^n.$$

Since $a \equiv 3 \pmod 4$, $\mathfrak{O}_k = \boldsymbol{Z}[\sqrt{-3a}]$, where $\mathfrak{O}_k$ is the principal order of $k$. Put $\omega = \sqrt{-3a}$. Let $\mathfrak{a}$ be an ideal of $\mathfrak{O}_k$ generated by $(4a^2 - 9)^{l'} + 2\omega$, then $((4a^2 - 9)^{l'} - 2\omega) = \bar{\mathfrak{a}}$. If $(\mathfrak{a}, \bar{\mathfrak{a}}) \neq 1$, then there exists a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \supseteq \mathfrak{a}$ and $\mathfrak{p} \supseteq \bar{\mathfrak{a}}$, hence $2(4a^2 - 9)^{l'} \in \mathfrak{p}$. As $2 \notin \mathfrak{p}$, $4a^2 - 9 \in \mathfrak{p}$. Moreover $\mathfrak{p} \supseteq \mathfrak{a}\bar{\mathfrak{a}} = (4a^2 + 9)^n$, then $4a^2 + 9 \in \mathfrak{p}$, hence $a \in \mathfrak{p}$, $3 \in \mathfrak{p}$. This contradicts $(a, 3) = 1$. Therefore $(\mathfrak{a}, \bar{\mathfrak{a}}) = 1$. Hence $\mathfrak{a}\bar{\mathfrak{a}} = (4a^2 + 9)^n$ means $\mathfrak{a} = \mathfrak{a}_0{}^n$, where $\mathfrak{a}_0$ is an ideal of $\mathfrak{O}_k$. By the assumption on the class number of $k$, $\mathfrak{a}_0$ is principal as $n$ is odd. The units of $\mathfrak{O}_k$ are $\{\pm 1\}$, thus

$$(4a^2 - 9)^{l'} + 2\omega = (\pm x \pm y\omega)^n,$$

where $x, y \in \boldsymbol{N}$, $4a^2 + 9 = x^2 + 3ay^2$. Since $y$ divides the imaginary part of $(\pm x \pm y\omega)^n$, $y$ divides 2, so $y = 1$ or 2.

Case (i-1) $y = 1$ : By $4a^2 + 9 = x^2 + 3a$

$$9a = (x + 2a - 3)(x - 2a + 3).$$

If $(x + 2a - 3, x - 2a + 3) \neq 1$, then a prime $p$ divides both $x + 2a - 3$ and $x - 2a + 3$. Hence we have $p \mid 2(2a - 3)$, $p^2 \mid 9a$. This contradicts the assumption $(a, 3) = 1$. Thus $(x + 2a - 3, x - 2a + 3) = 1$. As $x + 2a - 3 > 2a$, hence $x + 2a - 3 = 9a_1$, where $a = a_1a_2$ with $(a_1, a_2) = 1$. Then $9a_1 > 2a = 2a_1a_2$, so $9 > 2a_2$. Since $a_2$ is odd with $(a_2, 3) = 1$, we have $a_2 = 1$. Therefore $2(2a - 3) = 9a - 1$, that is, $a = -1$, which is a contradiction. Thus case (i-1) does not occur.

Case (i-2) $y = 2$: By $4a^2 + 9 = x^2 + 12a$, we have $x = 2a - 3$. Thus

$$(4a^2 - 9)^{l'} + 2\omega = (\pm(2a - 3) \pm 2\omega)^n$$
$$= \sum_{j=0}^{(n-1)/2}\binom{n}{n-2j}(\pm(2a - 3))^{n-2j}(-12a)^j$$

$$\pm 2\omega \sum_{j=0}^{(n-1)/2} \binom{n}{n-2j-1} (\pm(2a-3))^{n-2j-1}(-12a)^j.$$

So

$$\pm(2a-3)^{l'-1}(2a+3)^{l'} = \sum_{j=0}^{(n-1)/2} \binom{n}{n-2j} (2a-3)^{n-2j-1}(-12a)^j.$$

By Lemma 2, which we prove later, we have $l'=1$, i.e. $l=2$. Therefore

$$(4a^2-9)^2+12a=(4a^2+9)^n.$$

Such $n$ does not exist. Thus (i-2) does not occur, either.

Case (ii) $a \equiv 1 \pmod 4$ : Since $a \equiv 1 \pmod 4$, $a \geq 5$ and $\mathfrak{O}_k = \boldsymbol{Z}[(1+\omega)/2]$, where $\omega = \sqrt{-3a}$. From (1)

$$((4a^2-9)^{l'}+2\omega)((4a^2-9)^{l'}-2\omega)=(4a^2+9)^n.$$

By the assumption on the class number, we have easily, noticing that the ideal in $\mathfrak{O}_k$ generated by $(4a^2-9)^{l'}+2\omega$ can not be divisible by prime factor of 2,

(5) $$(4a^2-9)^{l'}+2\omega=((\pm x \pm y\omega)/2)^n$$

where $x, y \in \boldsymbol{N}$, $x \equiv y \pmod 2$, $4a^2+9=(x^2+3ay^2)/4$. From (5) we have

(6) $$2^n(4a^2-9)^{l'}+2^{n+1}\omega=(\pm x \pm y\omega)^n.$$

Since $y$ divides the imaginary part of $(\pm x \pm y\omega)^n$, $y$ divides $2^{n+1}$, so $y=1$ or $y=2^t$, $(t \in \boldsymbol{N})$.

Case (ii-1) $y=1$: Then $x$ is odd and $16a^2+36=x^2+3a$. Thus

(7) $$45a=(x+4a-6)(x-4a+6).$$

If $(x+4a-6, x-4a+6) \neq 1$, then a prime $p$ divides both $x+4a-6$ and $x-4a+6$. Hence we have $p \mid 4(2a-3)$, $p^2 \mid 45a$. Since $(a, 3)=1$, this is a contradiction. Hence $(x+4a-6, x-4a+6)=1$. Now $x+4a-6>4a$. Put $c=x+4a-6$. Then there are six possibilities on choice of $c$ in (7): (7.1) $c=45a$, (7.2) $c=9a$, (7.3) $c=5a$, (7.4) $c=9a_1$, (7.5) $c=5a_1$, (7.6) $c=45a_1$, where $a=a_1a_2$ with $(a_1, a_2)=1$, $a_2 \neq 1$.

As $x-4a+6=45a/c$, $8a-12=c-45a/c$. Hence (7.1)-(7.3) contradict $a \geq 5$. As $c>4a=4a_1a_2$, and $a_2$ is odd, neither (7.4) nor (7.5) occurs. In case (7.6), as $(a_2, 3)=1$, $a_2=5,7$ or $11$. No one of these satisfies $8a_1a_2-12=45a_1-a_2$ with integer $a_1$. Thus (ii-1) does not occur.

Case (ii-2) $y=2^t$, $t \in \boldsymbol{N}$. As $x$ is even, put $x=2x_0$. Then $4a^2+9=x_0^2+3\cdot 4^{t-1}a$. Assume $t \geq 2$, then $x_0$ is odd. If $x_0=1$, $a^2-3\cdot 4^{t-2}a+2=0$. As $a$ is odd, $t=2$, $a=1$, which is a contradiction. $(a, 3)=1$, then $x_0 \neq 3$. We put $x_0=3+2u$, $u \in \boldsymbol{N}$. Then $a^2=u^2+3u+3\cdot 4^{t-2}a$, so $1 \equiv 4^{t-2} \pmod 2$. Hence $t=2$, $4a^2+9=x_0^2+12a$. In (i-2) we have proved this is impossible. Thus $t=1$, $y=2$, so $4a^2+9=x_0^2+3a$. This case is treated in (i-1) and is also impossible. Thus (ii-2) does not occur, either. Hence $n$ is even. Then $m \neq 1$. Therefore $(l, m, n)=(2, 2, 2)$. (Cf. [1] Prop. 1, Theorem 1.)

**Lemma 2.** *Let $a \geq 5$ be odd with $(a, 3)=1$ and $n$ odd with $2l'>n \geq 3$. If*

(8) $$\pm(2a-3)^{l'-1}(2a+3)^{l'} = \sum_{j=0}^{(n-1)/2} \binom{n}{n-2j} (2a-3)^{n-2j-1}(-12a)^j,$$

*then* $l' = 1$.

*Proof.* Assume $l' \neq 1$. Put $2a - 3 = c$.

$$\pm c^{l'-1}(2a+3)^{l'} = c^2 \sum_{j=0}^{(n-3)/2} \binom{n}{n-2j} c^{n-2j-3}(-12a)^j + n(-12a)^{(n-1)/2},$$

so $c \mid n(-12a)^{(n-1)/2}$. $(c, a) = (c, 6) = 1$, we have $c \mid n$. $2l' > n \geq c \geq 7$, we have $l' \geq 4$. By induction on $t$, we shall prove that $c^t$ divides $n$, where $t$ is any odd integer. As such $n$ does not exist. $l'$ must be 1.

Let $t \geq 3$ and $c^{t-2} \mid n$. $2l' > n \geq c^{t-2} \geq 2t+1$, hence left hand side of (8) is divisible by $c^t$. Let $s$ be odd with $3 \leq s \leq t$. Now

$$\binom{n}{s} = \frac{n}{s}\binom{n-1}{s-1}, \quad \binom{n-1}{s-1} \in \boldsymbol{N}.$$

If $(c, s) = 1$, $c^{t-2} \left| \binom{n}{s} \right.$. Hence, $c^t \left| \binom{n}{s} c^{s-1} \right.$. If $(c, s) \neq 1$, let $s = s_0 \Pi_i p_i^{e_i}$, where each $p_i$ is a prime divisor of $c$ and $(c, s_0) = 1$. As $s - 1 - e_i \geq 2$, $c^2$ divides $c^{s-1}/\Pi p_i^{e_i}$. Hence $c^t \left| \binom{n}{s} c^{s-1} \right.$. Thus $c^t$ divides all terms of (8) except $n(-12a)^{(n-1)/2}$, hence $c^t \mid n$.

## References


[1] K. Takakuwa and Y. Asaeda: On a conjecture of Pythagorean numbers. Proc. Japan Acad., **69**A, 252–255 (1993).

[2] H. Wada and M. Saito: A table of ideal class groups of imaginary quadratic fields. Sophia Kokyuroku in Math., **28**, Sophia Univ. Press (1988).