53. *On the Order of Strongly Starlikeness of Strongly Convex Functions*

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1. Introduction. Let A denote the set of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that are analytic in $E = \{z : |z| < 1\}$. A function $f(z) \in A$ is called strongly starlike of order β, $0 < \beta \leqq 1$, if $|\arg(zf'(z)/f(z))| < \pi\beta/2$ in E.

Let us denote $\mathrm{STS}(\beta)$ the class of all functions which satisfy the above conditions. On the other hand, a function $f(z) \in A$ is called strongly convex of order β, $0 < \beta \leqq 1$, if $|\arg(1 + zf''(z)/f'(z))| < \pi\beta/2$ in E.

Let us denote $\mathrm{STC}(\beta)$ the class of all functions which satisfy the above conditions.

Mocanu [1, Corollary 1] obtained the following result.

If $f(z) \in A$ and

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi\gamma}{2} \text{ in } E,$$

then

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi\beta}{2}$$

where

$$\mathrm{Tan}\,\frac{\pi\gamma}{2} = \mathrm{Tan}\,\frac{\pi\beta}{2} + \frac{\beta}{(1-\beta)\cos\frac{\pi\beta}{2}}\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}$$

and $0 < \beta < 1$.

In this paper, we will prove the following theorem.

Main theorem. Let $f(z) \in \mathrm{STC}(\alpha(\beta))$. Then we have $f(z) \in \mathrm{STS}(\beta)$, where

$$\alpha(\beta) = \beta + \frac{2}{\pi}\mathrm{Tan}^{-1}\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{p(\beta) + \beta q(\beta)\cos\frac{\pi}{2}(1-\beta)}$$

$$p(\beta) = (1+\beta)^{\frac{1+\beta}{2}} \text{ and } q(\beta) = (1-\beta)^{\frac{\beta-1}{2}}.$$

2. Preliminaries. To prove the main theorem, we need the following lemma.

Lemma. *Let $p(z)$ be analytic in E, $p(0) = 1$, $p(z) \neq 0$ in E and suppose that there exists a point $z_0 \in E$ such that*

$$|\arg p(z)| < \frac{\pi\alpha}{2} \quad \textit{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\alpha}{2}$$

where $0 < \alpha$. *Then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \arg p(z_0) = \frac{\pi\alpha}{2}$$

and

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \textit{when} \arg p(z_0) = -\frac{\pi\alpha}{2}$$

where

$$p(z_0)^{1/\alpha} = \pm ia, \quad \textit{and } a > 0.$$

Proof. Let us put

$$q(z) = p(z)^{1/\alpha}.$$

Then we have

$$\operatorname{Re} q(z) > 0 \text{ for } |z| < |z_0|$$

and

$$\operatorname{Re} q(z_0) = 0.$$

Let us put $q(z_0) = \pm ia$, $a > 0$ and applying Nunokawa's result [2], we have

$$\frac{z_0 q'(z_0)}{q(z_0)} = \frac{1}{\alpha}\frac{z_0 p'(z_0) p(z_0)^{\frac{1}{\alpha}-1}}{p(z_0)^{1/\alpha}} = \frac{1}{\alpha}\frac{z_0 p'(z_0)}{p(z_0)} = ik$$

where k is a real and

$$k \geqq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{for } q(z_0) = ia$$

and

$$k \leqq -\frac{1}{2}\left(a + \frac{1}{a}\right) \quad \text{for } q(z_0) = -ia.$$

3. Proof of the main theorem. Let us put $p(z) = zf'(z)/f(z)$ and $f(z) \in \mathrm{STC}(\alpha(\beta))$.

If there exists a point $z_0 \in E$ such that

$$|\arg p(z)| < \frac{\pi\beta}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi\beta}{2}, \quad (0 < \beta < 1).$$

Putting

$$q(z) = p(z)^{1/\beta},$$

then we have

$$\operatorname{Re} q(z) > 0 \text{ for } |z| < |z_0|,$$

$$\operatorname{Re} q(z_0) = 0 \text{ and } q(z_0) = \pm ia$$

where a is a positive real number.

Then, from the lemma, we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\beta k$$

where k is a real number

$$k \geqq \frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when } q(z_0)=ia$$

and

$$k \leqq -\frac{1}{2}\left(a+\frac{1}{a}\right) \quad \text{when } q(z_0)=-ia$$

where $q(z_0)=p(z_0)^{1/\beta}=\pm ia$ and a is a positive real number.

At first, let us suppose $q(z_0)=ia$, $a>0$, then we have

$$1+\frac{z_0f''(z_0)}{f'(z_0)}=p(z_0)+\frac{z_0p'(z_0)}{p(z_0)}$$

$$=p(z_0)\left(1+\frac{z_0p'(z_0)}{p(z_0)^2}\right)=(ia)^\beta\left(1+i\beta k\frac{1}{(ia)^\beta}\right)$$

$$=a^\beta e^{i\frac{\pi\beta}{2}}\left\{1+e^{i\frac{\pi}{2}(1-\beta)}\beta k\frac{1}{a^\beta}\right\}$$

where $k\geqq\frac{1}{2}\left(a+\frac{1}{a}\right)$.

Then we have

$$\beta k\frac{1}{a^\beta}\geqq\frac{\beta}{2}(a^{1-\beta}+a^{-1-\beta}).$$

Let us put

$$g(a)=\frac{1}{2}(a^{1-\beta}+a^{-1-\beta}) \text{ and } a>0.$$

Then, by easy calculation, we have

$$g'(a)=\frac{1}{2}\{(1-\beta)a^{-\beta}-(1+\beta)a^{-2-\beta}\},$$

and $g(a)$ takes the minimum value at $a=\sqrt{(1+\beta)/(1-\beta)}$.

Therefore, we have

$$\arg\left(1+\frac{z_0f''(z_0)}{f'(z_0)}\right)=\arg p(z_0)+\arg\left(1+\frac{z_0p'(z_0)}{p(z_0)^2}\right)$$

$$=\frac{\pi\beta}{2}+\arg\left(1+e^{i\frac{\pi}{2}(1-\beta)}\frac{\beta k}{a^\beta}\right)$$

$$\geqq\frac{\pi\beta}{2}+\mathrm{Tan}^{-1}\frac{\left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}\sin\frac{\pi}{2}(1-\beta)}{1+\left(\frac{\beta}{1-\beta}\right)\left(\frac{1-\beta}{1+\beta}\right)^{\frac{1+\beta}{2}}\cos\frac{\pi}{2}(1-\beta)}$$

$$=\frac{\pi\beta}{2}+\mathrm{Tan}^{-1}\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{p(\beta)+\beta q(\beta)\cos\frac{\pi}{2}(1-\beta)}.$$

This contradicts the assumption of the main theorem.

For the case $q(z_0)=-ia$, $a>0$, applying the same method as the above, we have

$$\arg\left(1+\frac{z_0 f''(z_0)}{f'(z_0)}\right) \leqq -\frac{\pi\beta}{2} - \mathrm{Tan}^{-1}\frac{\beta q(\beta)\sin\frac{\pi}{2}(1-\beta)}{p(\beta)+\beta q(\beta)\cos\frac{\pi}{2}(1-\beta)}.$$

This contradicts the assumption. Therefore we complete the proof.

Putting $\beta = 0.5$ in the main theorem, we have the following result: If $f(z) \in \mathrm{STC}(\alpha(0.5))$, then we have

$$\left|\arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{4} \quad \text{in } E$$

where

$$\alpha(0.5) = 0.5 + \frac{2}{\pi}\mathrm{Tan}^{-1}\frac{1}{108^{1/4}+1}$$

$$\fallingdotseq 0.648.$$

References

[1] P. T. Mocanu: Alpha-convex integral operator and strongly starlike functions. Studia Univ. Babeş-Bolyai Mathematica, **34**, 2, 18–24 (1989).

[2] M. Nunokawa: On properties of non-Carathéodory functions. Proc. Japan Acad., **68**A, 152–153 (1992).