# 52. *Large-time Existence of Surface Waves of Compressible Viscous Fluid*

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**1. Introduction and theorem.** In this communication we are concerned with free boundary problem for compressible viscous isotropic Newtonian fluid which is formulated as follows: Find the domain $\Omega_t \subset \boldsymbol{R}^3$ occupied by the fluid at the moment $t > 0$ together with the density $\rho(x, t)$, velocity vector field $v(x, t) = (v_1, v_2, v_3)$ and the absolute temperature $\theta(x, t)$ satisfying the system of Navier-Stokes equations

$$(1.1)\quad \begin{cases} \dfrac{D\rho}{Dt} + \rho(\nabla\cdot v) = 0, \quad \rho\dfrac{Dv}{Dt} = \nabla\cdot\boldsymbol{P} - \rho g e_3, \\ \rho c_V \dfrac{D\theta}{Dt} + \theta p_\theta(\nabla\cdot v) = \nabla\cdot(\kappa\nabla\theta) + \Psi \\ (x \in \Omega_t \equiv \{x' = (x_1, x_2) \in \boldsymbol{R}^2, -b(x') < x_3 < F(x', t)\}, t > 0) \end{cases}$$

and the initial and boundary conditions

$$(1.2)\quad \begin{cases} (\rho, v, \theta)|_{t=0} = (\rho_0, v_0, \theta_0) \quad (x \in \Omega_0), \\ \boldsymbol{P}n = -p_e n + \sigma H n, \quad \kappa\nabla\theta\cdot n = \kappa_e(\theta_e - \theta) \\ (x \in \Gamma_t \equiv \{x' \in \boldsymbol{R}^2, x_3 = F(x', t)\}, t > 0), \\ v = 0, \ \theta = \theta_a \quad (x \in \Sigma \equiv \{x' \in \boldsymbol{R}^2, x_3 = -b(x')\}, t > 0), \\ \dfrac{D}{Dt}(x_3 - F) = 0 \quad (x \in \Gamma_t, t > 0), \quad F|_{t=0} = F_0(x') \quad (x' \in \boldsymbol{R}^2). \end{cases}$$

Here $\nabla = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}, \dfrac{\partial}{\partial x_3}\right)$; $\nabla' = (\nabla_1, \nabla_2) = \left(\dfrac{\partial}{\partial x_1}, \dfrac{\partial}{\partial x_2}\right)$; $\dfrac{D}{Dt} = \dfrac{\partial}{\partial t} + (v\cdot\nabla)$ is the material derivative; $\boldsymbol{P} = (-p + \mu'(\nabla\cdot v))\boldsymbol{I} + 2\mu\boldsymbol{D}(v) \equiv -p\boldsymbol{I} + \boldsymbol{V}$ is the stress tensor; $\boldsymbol{I}$ is the $3 \times 3$ unit matrix; $\boldsymbol{D}(v)$ is the veloctiy deformation tensor with the elements $D_{ij} = \dfrac{1}{2}\left(\dfrac{\partial v_i}{\partial x_j} + \dfrac{\partial v_j}{\partial x_i}\right)$; $\Psi = \mu'(\nabla\cdot v)^2 + 2\mu\boldsymbol{D}(v) : \boldsymbol{D}(v)$ is the dissipation function; $p = p(\rho, \theta)$ is the pressure with $p_\rho, p_\theta > 0$; $(\mu, \mu', \kappa, c_V)(\rho, \theta)$ are, respectively, coefficient of viscosity, second coefficient of viscosity, coefficient of heat conductivity, heat capacity at constant volume, which are all assumed to be known smooth functions of $(\rho, \theta)$ satisfying $\mu, \kappa, c_V > 0$, $2\mu + 3\mu' \geq 0$; $(g, \sigma, p_e, \kappa_e)$ are, respectively, acceleration of gravity, coefficient of surface tention, atmospheric pressure, coefficient of outer heat conductivity, which are all assumed to be positive constants; $e_3 = {}^t(0, 0, 1)$; $n = \dfrac{1}{\sqrt{1 + |\nabla' F|^2}}\ {}^t(-\nabla_1 F,$

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$-\nabla_2 F,\ 1)$ is the exterior unit normal vector to $\Gamma_t$; $H=\nabla'\cdot\left(\frac{\nabla' F}{\sqrt{1+|\nabla' F|^2}}\right)$ is the twice mean curvature of $\Gamma_t$.

We seek a solution of the problem (1.1)-(1.2) near the equilibrium rest state $(\rho, v, \theta, F)=(\bar{\rho}, 0, \bar{\theta}, 0)$, where $\bar{\theta}$ is any positive constant and $\bar{\rho}=\bar{\rho}(x_3)$ is determined by

$$(1.3)\qquad \int_{\bar{\rho}(0)}^{\bar{\rho}(x_3)} \frac{p_\rho(\eta, \bar{\theta})}{\eta}\, d\eta + g x_3 = 0,\quad p(\bar{\rho}(0), \bar{\theta}) = p_e.$$

We rewrite the problem (1.1)-(1.2) by changing the unknown functions $(\rho, v, \theta, F) \to (\rho+\bar{\rho}, v, \theta+\bar{\theta}, F)$ and by using (1.3) as follows:

(1.4)

$$\begin{cases} \rho_t + (v\cdot\nabla)(\rho+\bar{\rho}) + (\rho+\bar{\rho})(\nabla\cdot v) = 0, \\ (\rho+\bar{\rho})\,(v_t + (v\cdot\nabla)v) = \nabla\cdot \boldsymbol{V} - p_\rho \nabla\rho - p_\theta\nabla\theta + \left(\frac{\bar{\rho}}{\bar{p}_\rho}(p_\rho - \bar{p}_\rho) - \rho\right) g e_3, \\ (\rho+\bar{\rho}) c_V\,(\theta_t + (v\cdot\nabla)\theta) + (\theta+\bar{\theta}) p_\theta (\nabla\cdot v) = \nabla\cdot(\kappa\nabla\theta) + \Psi \ (x\in\Omega_t,\ t>0), \end{cases}$$

(1.5)

$$\begin{cases} (\rho, v, \theta)\,|_{t=0} = (\rho_0, v_0, \theta_0)(x) \quad (x\in\Omega_0), \\ 2\mu\Pi\boldsymbol{D}(v) = 0, \quad -(p-p_e) + \boldsymbol{V}n\cdot n = \sigma H, \\ \kappa\nabla\theta\cdot n = \kappa_e(\theta_e - \theta) \quad (x\in\Gamma_t,\ t>0),\ v=0,\ \theta=\theta_a \quad (x\in\Sigma,\ t>0), \\ F_t + v_1\nabla_1 F + v_2\nabla_2 F - v_3 = 0 \quad (x\in\Gamma_t,\ t>0), \quad F|_{t=0} = F_0(x') \quad (x'\in\boldsymbol{R}^2). \end{cases}$$

where $p = p(\rho+\bar{\rho}, \theta+\bar{\theta})$, $\bar{p}_\rho = p_\rho(\bar{\rho}, \bar{\theta})$ *etc.*, and $\Pi\varphi = \varphi - n(n\cdot\varphi)$.

Let $W_2^l(\Omega)\,(l>0,\ \Omega\subset\boldsymbol{R}^n)$ be the S. L. Sobolev-L. N. Slobodetskiĭ spaces. We denote the anisotropic spaces $W_2^{l,l/2}(Q_T)\,(Q_T=\Omega\times(0, T))$ of functions defined on $Q_T$ by $L_2(0, T; W_2^l(\Omega)) \cap L_2(\Omega; W_2^{l/2}(0, T))$.

Transforming the problem to the initial domain $\Omega_0$ by the relation

$$(1.6)\qquad x = \xi + \int_0^t \hat{v}(\xi, \tau)\, d\tau \equiv x(\xi, t),$$

where $\hat{v}(\xi, t)$ is the velocity vector field in Lagrangean coordinate system, we can establish temporarily local solvability of the problem (1.4)-(1.5) in the same way as in [ 4 ].

**Theorem 1.1** (local existence). *Let $b\in W_2^{5/2+l}(\boldsymbol{R}^2)$ with $l\in(1/2, 1)$. For arbitrary $\rho_0, v_0, \theta_0 \in W_2^{2+l}(\Omega_0)$, $F_0\in W_2^{7/2+l}(\boldsymbol{R}^2)$, $\theta_e\in W_2^{4+l,2+l/2}(\boldsymbol{R}_T^3)$, $\theta_a\in W_2^{5/2+l,5/4+l/2}(\Sigma_T)$, satisfying $\rho_0+\bar{\rho}>0$, $\theta_0+\bar{\theta}>0$, $\theta_e+\bar{\theta}>0$, $\theta_a+\bar{\theta}>0$ and the natural compatibility conditions (we omit them here) the problem* (1.4)-(1.5) *in Lagrangean coordinate system has the unique solution $(\hat{\rho}, \hat{v}, \hat{\theta})(\xi, t)$ defined on $Q_{T_1}\equiv\Omega_0\times(0, T_1)$ for some $T_1\in(0, T)$ such that $\hat{\rho}\in W_2^{2+l,1+l/2}(Q_{T_1})$, $\hat{v}, \hat{\theta}\in W_2^{3+l,3/2+l/2}(Q_{T_1})$ and*

$$\begin{aligned} &\hat{E}^{3+l}(Q_{T_1}) \equiv \|\hat{\rho}\|_{W_2^{2+l,1+l/2}(Q_{T_1})} + \|(\hat{v}, \hat{\theta})\|_{W_2^{3+l,3/2+l/2}(Q_{T_1})} \\ (1.7)\qquad &\le c_1\,(\|\rho_0, v_0, \theta_0)\|_{W_2^{2+l}(\Omega_0)} + \|F_0\|_{W_2^{7/2+l}(R^2)} + \|\theta_e\|_{W_2^{4+l,2+l/2}(R_T^3)} \\ &\quad + \|\theta_a\|_{W_2^{5/2+l,5/4+l/2}(\Sigma_T)}) \equiv c_1 E_{0,T}. \end{aligned}$$

*The number $T_1$ increases unboundedly as $E_{0,T}$ tends to zero. Moreover, the solution possesses some additional regularity with respect to $t$:*

(1.8) $$\sup_{t_1<t<T_1} (\| \hat{\rho} \|_{W_2^{2+l}(\Omega_0)} + \| (\hat{v}, \hat{\theta}) \|_{W_2^{3+l}(\Omega_0)}) \leq c_2(E_{0,T} + \hat{E}^{3+l}(Q_{T_1}))$$

*for any* $t_1 > 0$, $t_1 < T_1$.

The following is our main theorem.

**Theorem 1.2** (global existence). *Under the assumptions of Theorem* 1.1, *if* $E_0 \equiv E_{0,\infty} \leq \varepsilon$ *with sufficiently small number* $\varepsilon$, *then the problem* (1.4)-(1.5) *has the unique solution* $(\rho, v, \theta, F)$ *for all* $t > 0$ *satisfying*

(1.9) $$\sup_{t \geq t_1} (\| \rho \|_{W_2^{2+l}(\Omega_t)} + \| (v, \theta) \|_{W_2^{3+l}(\Omega_t)} + \| F \|_{W_2^{7/2+l}(R^2)}) \leq c_3 E_0$$

*with each* $t_1 > 0$.

Similar result was established for barotropic fluid bounded only by a free surface in [ 3 ].

**2. Proof of Theorem 1.2.** Theorem 1.2 is proved by combination of the local existence theorem and the a priori estimate. To state the a priori estimate, it is convenient to make use of the coordinate transformation mapping from $\Omega_t$ onto the equilibrium domain $\bar{\Omega} \equiv \{y' \in \boldsymbol{R}^2, -b(y') < y_3 < 0\}$ defined by

(2.1) $$(x_1, x_2, x_3) = \left(y_1, y_2, \tilde{F} + y_3\left(1 + \frac{\tilde{F}}{b}\right)\right) \equiv x(y, t),$$

where $\tilde{F}$ is the extension of $F$ to $\bar{\Omega} \times \boldsymbol{R}_+$ (see [1]). Let us put $\tilde{f}(y, t) = f(x(y,t), t)$ and

$$\tilde{E}^{3+1}(\bar{Q}_T) \equiv \| \tilde{\rho} \|_{W_2^{2+l,1+l/2}(\bar{Q}_T)} + \| (\tilde{v}, \tilde{\theta}) \|_{W_2^{3+l,3/2+l/2}(\bar{Q}_T)} + \| F \|_{W_2^{7/2+l,7/4+l/2}(R_T^2)}, \quad \bar{Q}_T = \bar{\Omega} \times (0, T).$$

**Theorem 2.1** (a priori estimate). *Let* $(\rho, v, \theta, F)$ *be the solution of* (1.4)-(1.5) *defined on* $0 < t < T$. *If* $E_{0,T} < \varepsilon_1$ *and* $\tilde{E}^{3+1}(\bar{Q}_T) < \delta_1$ *with sufficiently small* $\varepsilon_1$, $\delta_1$, *then the following a priori estimate holds*:

(2.2) $$\tilde{E}^{3+1}(\bar{Q}_T) \leq c_4 E_{0,T}.$$

*Proof of Theorem* 1.2. Let $E_0$ be so small that the problem (1.4)-(1.5) in Lagrangean coordinate system is solvable on the interval (0,1). Such a solution satisfies inequalities (1.7), (1.8) for $T_1 = 1$. Furthermore, (2.2) with $T = 1$ is valid provided that $E_0 < \varepsilon_1$ and $c_1 E_0 < \delta_1$. Combining these inequalities, we find that $E_1 \leq c_5 E_0$ ($E_1$ is the norms of the data at $t = 1$). Introducing new Lagrangean coordinate system $\xi \in \Omega_1$ and again applying Theorem 1.1, we can establish the solvability of the problem for $t \in (1,2)$ provided that $E_0$ is sufficiently small. Repeating this process infinitely many times, we arrive at the assertion of the theorem.

**3. A priori estimate.** First we rewrite the system (1.4)-(1.5) so that all the nonlinear terms appear in the right hand side of equations and next make transformation to the equilibrium rest domain $\bar{\Omega}$ and linearize it again. Then we finally obtain

(3.1) $$\begin{cases} \tilde{\rho}_t + \bar{\rho}(\nabla \cdot \tilde{v}) + (\tilde{v} \cdot \nabla)\bar{\rho} = f_1, \\ \bar{\rho}\tilde{v}_t - \nabla \cdot \bar{V} + \bar{p}_\rho \nabla \tilde{\rho} + \bar{p}_\theta \nabla \tilde{\theta} - \left(\dfrac{\bar{\rho}}{\bar{p}_\rho} (dp_\rho)_{(\bar{\rho},\bar{\theta})}(\tilde{\rho}, \tilde{\theta}) - \tilde{\rho}\right) g e_3 = f_2, \\ \bar{\rho}\bar{c}_V \tilde{\theta}_t - \nabla \cdot (\bar{\kappa} \nabla \tilde{\theta}) + \bar{\theta}\bar{p}_\rho (\nabla \cdot \tilde{v}) = f_3 \ \textit{in} \ \bar{Q}_T, \end{cases}$$

$$
(3.2)\begin{cases}
(\tilde{\rho},\ \tilde{v},\ \tilde{\theta})\,|_{t=0} = (\tilde{\rho}_0,\ \tilde{v}_0,\ \tilde{\theta}_0)(y) \quad on\ \bar{\Omega}, \\
\bar{\mu}\left(\dfrac{\partial \tilde{v}_k}{\partial y_3} + \dfrac{\partial \tilde{v}_3}{\partial y_k}\right)\Big|_{y_3=0} = f_{3+k} \quad (k = 1,2), \\
-\ (dp)_{(\bar{\rho},\bar{\theta})}(\tilde{\rho},\ \tilde{\theta}) + \bar{\mu}'(\nabla\cdot\tilde{v}) + 2\bar{\mu}\dfrac{\partial \tilde{v}_3}{\partial y_3} - \sigma\nabla'^2 F - \bar{p}_0' F\Big|_{y_3=0} = f_6, \\
\bar{\kappa}\dfrac{\partial \tilde{\theta}}{\partial y_3} + \kappa_e\tilde{\theta}\Big|_{y_3=0} = \kappa_e\tilde{\theta}_e + f_7 \quad on\ \boldsymbol{R}_T^2, \\
\tilde{v} = 0, \quad \tilde{\theta} = \theta_a, \quad on\ \textstyle\sum_T, \\
F_t - \tilde{v}_3\,|_{y_3=0} = f_8 \quad on\ \boldsymbol{R}_T^2, \quad F\,|_{t=0} = F_0(y') \quad on\ \boldsymbol{R}^2,
\end{cases}
$$

where $\bar{V} = \bar{\mu}'(\nabla\cdot\ \tilde{v})I + 2\bar{\mu}\,\boldsymbol{D}(\tilde{v})$, $\bar{p}_0' = \dfrac{\partial}{\partial x_3}p(\bar{\rho}(x_3),\ \bar{\theta})\,|_{x_3=0}$ and $f = \{f_i\ (i = 1,\ldots,8)\}$ are at least quadratic functions of $(\tilde{\rho},\ \tilde{v},\ \tilde{\theta},\ \tilde{F})$ and their first and second derivatives. The estimate of the linearized problem (3.1)-(3.2) with given $f$ reads as follows.

**Lemma 3.1.** *Let* $b \in W_2^{3/2+l}$ *with* $l \in (1/2,1)$, $\tilde{\rho}_0, \tilde{v}_0, \tilde{\theta}_0 \in W_2^{1+l}(\bar{\Omega})$, $F_0 \in W_2^{5/2+l}(\boldsymbol{R}^2)$, $f_1 \in W_2^{1+l,1/2+l/2}(\bar{Q}_T)$, $f_2, f_3 \in W_2^{l,l/2}(\bar{Q}_T)$, $f_{3+k}, f_6, f_7 \in W_2^{1/2+l,1/4+l/2}(\boldsymbol{R}_T^2)$, $f_8 \in W_2^{3/2+l,3/4+l/2}(\boldsymbol{R}_T^2)$, $\theta_e \in W_2^{3+l,3/2+l/2}(\boldsymbol{R}_T^3)$, $\theta_a \in W_2^{3/2+l,3/4+l/2}(\sum_T)$ *and the compatibility conditions are satisfied. Then for the problem* (3.1)-(3.2), *we have the estimate*

$$
(3.3)\quad
\begin{aligned}
&\|\tilde{\rho}\|_{W_2^{1+l,1/2+l/2}(\bar{Q}_T)} + \|(\tilde{v},\ \tilde{\theta})\|_{W_2^{2+l,1+l/2}(\bar{Q}_T)} + \|F\|_{W_2^{5/2+l,5/4+l/2}(R_T^2)} \\
&\leq c_6\,(\|(\tilde{\rho}_0,\ \tilde{v}_0,\ \tilde{\theta}_0)\|_{W_2^{1+l}(\bar{\Omega})} + \|F_0\|_{W_2^{5/2+l}(R^2)} + \|f_1\|_{W_2^{1+l,1/2+l/2}(\bar{Q}_T)} \\
&\quad + \|(f_2,\ f_3)\|_{W_2^{l,l/2}(\bar{Q}_T)} + \|(f_{3+k},\ f_6,\ f_7)\|_{W_2^{1/2+l,1/4+l/2}(R_T^2)} \\
&\quad + \|f_8\|_{W_2^{3/2+l,3/4+l/2}(R_T^2)} + \|\theta_e\|_{W_2^{3+l,3/2+l/2}(R_T^3)} + \|\theta_a\|_{W_2^{3/2+l,3/4+l/2}(\Sigma_T)}).
\end{aligned}
$$

*Proof of Theorem* 2.1. We first apply (3.3) to the problem (3.1)-(3.2) and establish the a priori estimate for lower order terms. For the derivatives of highest order, we appeal to the energy method as in [2,3]. The details will be published elsewhere.

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