34. *Pseudo Volume Forms and their Applications to Holomorphic Mappings*

By Pei-Chu HU*) and Chung-Chun YANG**)

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1. A Generalization of Schwarz's lemma. Let M and N be complex manifolds of dimension m and n, respectively and $f: M \to N$ denote a holomorphic mapping. Let θ and ω be the associated 2-forms of hermitian metrics ds_M^2 and ds_N^2 on M and N, respectively. Let Φ be a non-negative (m, m)-form of class C^∞ on M and define a function u by

(1) $$\Phi = u\theta^m.$$

For a function λ on M, define

(2) $$E_\lambda = f^*(Ric\omega^n) - \lambda Ric\Phi.$$

If rank of $f \geq b > 0$ with u_b defined to be

(3) $$\Phi = u_b f^*(\omega^b) \Lambda\, \theta^{m-b}$$

then u can be estimated as follows.

Theorem 1.1. *Let M be a complete Kahler manifold with the Ricci curvature bounded from below and let N be a hermitian manifold with the Ricci curvature bounded from above by a negative constant K_2. Suppose the rank of $f \geq b > 0$. If there exist a constant K_1, a non-negative function λ bounded from above and a non-negative (m, m)-form $\Phi \not\equiv 0$ of class C^∞ such that*

$$\lambda R - Tr(E_\lambda) \geq K_1, \quad sup\, u_b < \infty,$$

where R is the scalar curvature of M, then $K_1 < 0$, and

$$0 < sup\, u \leq \binom{n}{b} \left(\frac{K_1}{bK_2}\right)^b sup\, u_b.$$

As consequences and applications of Theorem 1.1, we exhibit some special and wellknown cases as follows.

Special case 1. Suppose

$$m = n = b,\ \lambda = 1,\ \Phi = f^*(\omega^n).$$

Then $E_1 = 0$, $u_n = 1$. Hence we have $0 < sup\, u \leq \left(\frac{K_1}{nK_2}\right)^n$, which includes the results of Yau [8] and Chern [1].

Special case 2. Suppose

$$m > n = b,\ \lambda = 1,\ \Phi = i_{m-n} f^*(\omega^n) \wedge \varphi \wedge \bar{\varphi}$$

where φ is a holomorphic $(m-n)$-form on M. We can prove

$$E_1 = 0, \quad u_n \leq |\varphi|^2.$$

*) Department of Mathematics, Shandon University, China.

**) Department of Mathematics, The Hong Kong University of Science and Technology, Hong Kong.

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Hence we have

$$0 < sup\, u \leq \left(\frac{K_1}{nK_2}\right)^n sup\,|\,\varphi\,|^2.$$

Also if $M = C^m$, we can choose φ such that $sup\,|\,\varphi\,|^2 < \infty$. For the Euclidian metric on C^m, we have $R = 0 = K_1$.

Corollary 1.2. *If N is a hermitian manifold with Ricci curvature bounded from above by a negative constant, then any holomorphic mapping $f : C^m \to N$ has everywhere rank less than n.*

Kodaira [4] proved this Corollary when N is pseudo canonical.

Special case 3. Suppose

$$\Phi = f^*(\omega^B) \wedge \theta^{m-b}.$$

Then $u_b = 1$. Assume that $R \geq K$ (constant), and that λ is a positive constant. Then Theorem 1.1 implies

$$sup\, Tr(E_\lambda) > \lambda K.$$

Further if $M = C^m$ and if ds_M^2 is the Euclidian metric, then

$$sup\, Tr(E_\lambda) > 0.$$

Special case 4. If $n > m = b$ and if M is Stein, Stoll [6] constructed a pseudo volume form $\Phi = F^*[\omega^n]$, where F is an effective Jacobian section, such that $E_1 = 1$. For more detail on pseudo volume forms, see Lang [5].

2. A main formula for pseudo volume forms. Let M be a complex manifold of dimension m with a parabolic exhaustion function $\tau : M \to [0, \infty)$ and set

$$v = dd^c\tau, \quad \sigma = d^c \log\tau \wedge (dd^c \log\tau)^{m-1}.$$

For a subvariety A of pure dimension $k(\leq m)$ in M and $a(p, p)$-form x on M with $0 \leq p \leq k$, define

$$A(r, x) = r^{2p-2k}\int_{A[r]} x \wedge v^{k-p},\ A(r, s\,; x) = \int_s^r A(t\,; x)\frac{dt}{t}$$

where $A[r] = \{x \in A \mid \tau(x) \leq r^2\}$. Then

$$N(r, s, A) := A(r, s\,; 1) \quad (p = 0)$$

is just the valence function of A. For a non-negative function ρ on M, set

$$m(r\,; \rho) = \int_{\partial M[r]} (\log\rho)\sigma, \quad m(r, s\,; \rho) = m(r\,; \rho) - m\,(s\,; \rho).$$

Let ρ be a continuous function on M which is C^∞ outside a proper analytic subset D, and which locally in terms of complex coordinates can be expressed as

$$\rho(z) = h(z)\,|\,g(z)\,|^{2q} \tag{4}$$

where q is some fixed rational number > 0, h is in C^∞ and > 0, and g is holomorphic not identically zero. Then the following formula can be obtained

$$M(r, s\,; dd^c \log\rho) + qN(r, s, D) = m(r, s\,; \rho), \tag{5}$$

where $D = (g = 0)$ is the (zero) divisor of ρ, which implies FMT for divisors (see [3], [6]).

Let Ψ be a pseudo volume form on N of order q (see Lang [5]). Locally in terms of complex coordinates Ψ can be expressed as

$$\Psi(z) = \rho(z)\prod_{i=1}^n \frac{\sqrt{-1}}{2\pi} dz_i \wedge d\bar{z}_i$$

where $\rho(z)$ satisfies the properties (4). Let Ω be a volume form on N and define a function ζ by

(6) $$\Psi = \zeta\Omega.$$

Then (6) yields, if $f(M) \not\subseteq D_{\Psi}$,

(7) $$M(r, s; f^*(Ric\Psi)) + qN(r, sf^{-1}(D_{\Psi})) = M(r, s; f^*(Ric\Omega)) + m(r, s; \zeta \circ f).$$

Here D_{Ψ} is the zero divisor of Ψ. Let Ψ be a pseudo-volume form on M of order q_0 and define a function h on M by

(8) $$\Phi = hv^m.$$

Then from (7), we obtain

(9) $$M(r, s(Ric\Phi) + q_0N(r, s,(D_{\Phi})) = Ric_{\tau}(r, s) + m(r, s; h)$$

where $Ric_{\tau}(r, s)$ is the Ricci function of τ (see Stoll [6]). Hence the Stoll's formula ([6], Th. 15, 5) and Plucker Difference Formula (see Stoll [7]) follow from (9).

3. A generalization of a Kodaira-Griffiths theorem. We continue with the situation $f: M \to N$ of §2 where we assume that N is pseudo canonical (or general type). Here we set

(10) $$E_{\lambda} = f^*(Ric\Psi) - \lambda Ric\Phi.$$

Let L be a positive holomorphic line bundle on N and let $\omega > 0$ be the curvature form (or Chern form) of L for a hermitian metric in L. By Kodaira [4], Lang [5], there exist integers p and k such that L^p is very ample, and

$$P_k(L^p) := dim H^0(N, K_N^k \otimes L^{-p}) > 0$$

where K_N is the canonical line bundle on N. Let $B_{p,k}$ be the base locus of the linear system $H^0(N, K_N^k \otimes L^{-p})$ and let

$$B_p = \cap_k B_{p,k}$$

where the intersection extends over all k with $P_k(L^p) > 0$. As applications of the formulas (7) and (9), we obtain

Theorem 3.1. *Assume M, N, L, Ψ, Φ and f as above. Suppose that rank of $f \geq b > 0$ and define a function u_b by*

(11) $$\Phi = u_b f^*(\omega^b) \wedge v^{m-b}.$$

If $f(m) \not\subseteq B_p \cup D_{\Psi}$, then for $\lambda = 0$,

(12) $$\|_{\varepsilon} \left(\frac{p}{k} - o(1)\right)T(r, s, L) \leq \lambda Ric_{\tau}(r, s) + M(r, s; E_{\lambda}) + qN(r, s, f^{-1}(D_{\Psi})) - \lambda q_0 N(r, s, D_{\Phi}) + m(r; u_b^{\lambda}/\zeta \circ f) + c\varepsilon \log r$$

where $c > 0$ is a constant, while $T(r, s, L) = M(r, s; f^*(\omega))$, and where the notation $\|_{\varepsilon}$ means that the inequality holds except on an open set I_{ε} with $\int_{I_{\varepsilon}} r^{\varepsilon} dr < \infty$ for some $\varepsilon > 0$.

Let M be affine algebraic, and take

(13) $$\Psi = \omega^n,\ \Phi = i_{m-b} f^*(\omega^b) \wedge \varphi \wedge \bar{\varphi}$$

where φ is a holomorphic $(m-b)$-form on M. According to Griffiths-King [3] and Stoll [6] there exist a parabolic exhaustion τ on M and φ such that $\Phi \not\equiv 0$, $u_b \leq 1$, and

$$\lim_{r\to\infty} Ric_{\tau}(r, s) / \log r < \infty.$$

Hence Theorem 3.1 implies

Corollary 3.2. *Suppose that rank of* $f \geq b > 0$, *and* $f(M) \not\subseteq B_p$. *If* M *is affine algebraic, and if for some* $\lambda > 0$

$$e_\lambda(b) := \limsup_{r \to \infty} M(r, s; E_\lambda)/\log r < \infty,$$

then f *is rational.*

If $m \geq n = b =$ rank of f, then $E_1 = 0$. Hence Corollary 3.2 yields

Corollary 3.3 (Griffiths). *Let* M *be affine algebraic. Then any holomorphic mapping* $f: M \to N$ *whose image contains an open set is necessarily rational.*

Corollary 3.4 (Kodaira). *Any holomorphic mapping* $f: C^m \to N$ *has everywhere rank less than* n.

Corollary 3.5. *Take* $M = C^m$. *If rank of* $f \geq b > 0$ *and if* $e_\lambda(b) \leq 0$ *for some* $\lambda > 0$, *then* $f(C^m) \subseteq B_p$.

4. A generalization of Landau-Schottky theorem. Here we consider a holomorphic mapping $f: C^m(s) \to N$; a pseudo canonical variety N, where

$$C^m(s) = \{z = (z^1, \ldots, z^m) \in C^m \mid |z|^2 = \sum_{i=1}^m |z^i|^2 < s^2\}.$$

Define τ by $\tau(z) = |z|^2$, and take $\Psi = \Omega$ and $M = C^m(s)$. Also define h, u_b and E_λ by (8), (12), and (10), respectively.

Theorem 4.1. *Let* N *be a pseudo canonical variety, and* x_0 *a point on* N *such that* $\alpha(x_0) \neq 0$ *for an element* $\alpha \in H^0(N, K_N^k \otimes L^{-p})$. *Assume that* $f(0) = x_0$, $h(0) \geq 1$, *and that*

$$k = \sup u_b < \infty, \ M(r, 0; E_\lambda) \leq 0.$$

Then there exists a constant $R = R(b, k, p, \lambda, k)$ *with the following properties. For any holomorphic mapping* $f: C^m(s) \to N$ *with rank of* $f \geq b > 0$, *the inequality* $s \leq R$ *holds.*

Corollary 4.2. *Let* N *be a pseudo canonical variety, and* x_0 *point on* N *such that* $\alpha(x_0) \neq 0$ *for an element* $\alpha \in H^0(N, K_N^k \otimes L^{-p})$. *Then there exists an absolute constant* R *with the following properties: For any holomorphic mapping* $f: C^m(s) \to N$ *with* $f(0) = x_0$ *and* $h(0) \geq 1$, *the inequality* $s \geq R$ *holds, where* h *is defined by*

$$\Phi = i_{m-n} f^*(\Omega) \wedge \varphi \wedge \bar{\varphi} = hv^m$$

for some holomorphic $(m-n)$*-form* φ *on* $C^m(s)$.

Here $m \geq n =$ rank of f. If $m = n$, this corollary was proved by Kodaira [4]. Note that, Ω can be chosen so that $h(0)$ is just the Jacobian of f at the origin.

Corollary 4.3. *Let* $f: C^m \to N$ *be a holomorphic mapping from* C^m *to a pseudo canonical variety* N *with* $n > m =$ *rank of* f. *For an effective Jacobian section* F, *define a function* u_b *by*

$$F[\Omega] = u_b f^*(\omega^b) \wedge v^{m-b}.$$

If $\sup u_b < \infty$ *for some* b *with* $1 \leq b \leq m$, *then* $f(C^m) \subseteq B_p$.

Note that by using Theorem 3.1, when $m \geq 2$, the condition $\sup u_b < \infty$ in the corollary can be replaced by the following weak condition:

$$\|_\varepsilon \ m(r; u_b) \leq o(T(r, s, L)) + o(\log r).$$

Generally, if $m = 1$, $f(C)$ is contained in the Green-Griffiths set (see [2], [5]).

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