4. *Hasse's Norm Theorem for K_2*

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1. Introduction and definitions. In this note, we shall present a description of Galois groups of the quotient field of 2-dimensional local ring and Hasse principle for K_2 of such fields by using hypercohomology and Lichtenbaum's complex $\boldsymbol{Z}(2)$. This note is an announcement of author's doctor thesis [2].

Unless the contrary is explicitly stated, we shall employ the following notation throughout this paper: For a field K, K_s is a fixed separable closure of K. Let G be a group and M a G-module. We denote M^G by $\Gamma(G, M)$, which is viewed as a functor. The symbol $\boldsymbol{Z}(2)$ stands for Lichtenbaum's complex. For definitions and properties on Lichtenbaum's complex, see [3] and [4]. In this note we shall freely use the standard notations on complexes and objects in derived categories as in [3] and [4].

Let A be a two dimensional complete normal local ring whose residue field F is a finite field, K its quotient field and P the set of all prime ideals of A of height one. For each $\mathfrak{p} \in P$, let $A_\mathfrak{p}$ be the completion of the localization of A at $\mathfrak{p}$, $K_\mathfrak{p}$ its quotient field and $\kappa(\mathfrak{p})$ the residue field of $A_\mathfrak{p}$. Note that by [6], $K_\mathfrak{p}$ is a two dimensional local field and $\kappa(\mathfrak{p})$ is a local field in the usual sense.

We shall construct the complex which represents K_2-idele class group, which is defined in [6]. We define first an auxiliary complex. Under the above notation, let $L_\mathfrak{P}$ be a finite unramified extension of $K_\mathfrak{p}$, where $\mathfrak{P}$ is a prime above $\mathfrak{p}$. Then the complex $Q(L_\mathfrak{P})$ [1] is defined to be the mapping cone of the following morphism of complexes:

$$\tau_{\leq 2}\boldsymbol{R}\Gamma(H_\mathfrak{p}, \boldsymbol{Z}(2)) \rightarrow F(\mathfrak{p})^\times[-2],$$

where $H_\mathfrak{p} = \mathrm{Gal}((K_s)_\mathfrak{p}/L_\mathfrak{P})$ and $F(\mathfrak{p})$ is the residue field of $L_\mathfrak{P}$.

We also define K_2-idele complex. Let L be a finite extension of K. The complex $I(L)$ is defined as follows. First we set

$$I^S(L) = \prod_{\mathfrak{p}\in S} \tau_{\leq 2}\boldsymbol{R}\Gamma(H_\mathfrak{p}, \boldsymbol{Z}(2)) \times \prod_{\mathfrak{p}\in P-S} Q(L_\mathfrak{P}),$$

for a finite subset S of P containing all the ramified primes in L/K. Then the $I(L)$ is defined by

$$I(L) = \varinjlim_{S} I^S(L).$$

The idele complex $\boldsymbol{I}_K$ is defined as

$$\boldsymbol{I}_K = \varinjlim_{L} I(L),$$

where the limit runs through all finite extensions of K.

Now we can define our K_2-idele class complex. The complex $C(L)$ is defined by the mapping cone of the following morphism of complexes:

$$\tau_{\leq 2}\boldsymbol{R\Gamma}(H, \boldsymbol{Z}(2)) \to I(L),$$

where $H = \mathrm{Gal}(K_s/L)$. And the "idele class complex" $\boldsymbol{C}_K$ is defined as follows:

$$\boldsymbol{C}_K = \varinjlim_{L} C(L).$$

Remark. We work in the category of complexes of G-modules. So in general the mapping cone are not canonically defined. But in our case we can construct $C(L)$ in the category of complexes of G-modules, and our construction is canonical in the category of complexes.

Our K_2-idele complex and K_2-idele class complex have the following properties.

Proposition 1. (1) *$\boldsymbol{C}_K$ is acyclic outside* [1, 2]

(2) $$H^2(\mathrm{Gal}(K_s/K), \boldsymbol{C}_K) = C_K.$$

Here C_K is a K_2-idele class group.

Proposition 2. $H^3(K, \boldsymbol{I}_K) = 0.$

2. Hasse principle. The aim of this section is to give an expression of $\mathrm{Gal}(L/K)^{ab}$ under some special conditions and prove Hasse's norm theorem for K_2 as one of application of modified hypercohomology, which is defined in [1] and is denoted here by $\hat{H}^q(G, *)$. The following theorem is our main result from technical viewpoint.

Theorem 3. *Let L/K be a finite Galois extension such that the integral closure of A in the extension field L is regular. Then the finite group* $\mathrm{Gal}(L/K)$ *and the complex of* $\mathrm{Gal}(L/K)$*-module* $\tau_{\leq 0}\boldsymbol{R\Gamma}(\mathrm{Gal}(K_s/L), \boldsymbol{C}_K[2])$ *satisfy the assumptions of the generalized Tate-Nakayama theorem.*

From the previous theorem we get familiar description of $\mathrm{Gal}(L/K)$. Namely, by using the next theorem, we have the Corollary 5.

Theorem 4 (Generalized Tate-Nakayama theorem) [1, Thm.2.1]. *Let G be a finite group, $A^{\cdot}$ a complex of G-modules such that except A^0 and $A^{(-1)}$ all terms are zero. Let a be an element of $\hat{H}^2(G, A^{\cdot})$. Assume that for each p-Sylow subgroup G_p of G:*

(1) $$\hat{H}^1(G_p, A^{\cdot}) = 0.$$

(2) *$\hat{H}^2(G_p, A^{\cdot})$ is generated by $\mathrm{Res}_{G/G_p}(a)$ whose order is equal to $|G_p|$.*

Then, for all $q \in \boldsymbol{Z}$ and all subgroup H of G,

$$\hat{H}^q(H, A^{\cdot}) \simeq \hat{H}^{q-2}(H, \boldsymbol{Z}).$$

Corollary 5. *Let L/K be a finite Galois extension of K. Assume that the integral closure of A in the field L is regular. Then we have the following isomorphism:*

$$\mathrm{Gal}(L/K)^{ab} \simeq C_K/N_{L/K}C_L.$$

The proof of Thm. 3 can be reduced to the following Lemma 6, as in the case of classical class field theory.

Lemma 6. *Let L/K be a finite Galois extension of K and M be any intermediate field of L/K. And assume that the integral closure of A in the field L is regular.*

(1) *For all integers $q > 2$, we have $H^q(M, \boldsymbol{C}_M[2]) = 0$.*

(2) $$H^1(M, \boldsymbol{C}_M[2]) = 0.$$

(3) *There is an isomorphism* $\mathrm{inv}_M : H^2(M, \boldsymbol{C}_M[2]) \to \boldsymbol{Q}/\boldsymbol{Z}$.

(4) *The following diagram is commutative;*

$$\begin{array}{ccc} H^2(M, \boldsymbol{C}_M[2]) & \xrightarrow{\mathrm{Res}} & H^2(N, \boldsymbol{C}_N[2]) \\ \downarrow & & \downarrow \\ \boldsymbol{Q}/\boldsymbol{Z} & \xrightarrow{n} & \boldsymbol{Q}/\boldsymbol{Z} \end{array}$$

where M and N are intermediate fields of L/K such that $N \supset M$ and $[N : M] = n$.

In the proof of the above Lemma 6, we use Saito's Hasse principle in [7], which plays an important role.

As another application of modified hypercohomology, we obtain the "Hasse Principle in Relative Case".

Proposition 7. *Let L/K be a finite Galois extension of K. Assume that the integral closure of A in the field L is regular. And let $\mathfrak{P}$ be a prime ideal of height one in the integral closure of A in L which is lying over $\mathfrak{p}$. Then the following sequence is exact;*

$$0 \to \hat{H}^2(L/K, T) \to \bigoplus_{\mathfrak{p} \in P} \frac{1}{[L_{\mathfrak{P}} : K_{\mathfrak{p}}]} \boldsymbol{Z}/\boldsymbol{Z} \xrightarrow{\sigma} \frac{1}{[L : K]} \boldsymbol{Z}/\boldsymbol{Z} \to 0,$$

where $T = \tau_{\leq 0} \boldsymbol{R}\Gamma(H, \boldsymbol{Z}(2)[2])$ and $H = \mathrm{Gal}(K_s/L)$.

Corollary 8. *Let L/K be a cyclic extension of K. Assume that the integral closure of A in the field L is regular. Let x be an element of K_2K. If for each $\mathfrak{p}$ the diagonal image of x is contained in $N_{L_{\mathfrak{P}}/K_{\mathfrak{p}}} K_2 L_{\mathfrak{p}}$, then $x \in N_{L/K} K_2 L$.*

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