27. *A Partial Criterion for Ergodicity of Geodesic Flows on Surfaces with Infinite Area and Negative Curvature*

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Abstract: Our problem is under what conditions geodesic flows on surfaces with infinite area and non-constant negative curvature are ergodic. Our result is that ergodicity is preserved under the change of the non-Euclidean metric on any compact set.

1. Criteria in the case of constant negative curvature. Let $U = \{(x, y) \in R^2 ; x^2 + y^2 = 1\}$ be the unit circle and $D = \{(x, y) \in R^2 ; x^2 + y^2 < 1\}$ be the interior of U. We give on D a non Euclidean metric

$$dc^2 = 4(dx^2 + dy^2) / (1 - x^2 - y^2)^2.$$

We give a discrete subgroup Γ of linear fractional transformations which map D to D and U to U. Let $M = D/\Gamma$. Then we have a 2-dimensional Riemannian manifold (M, c). We denote by d_c the distance of M introduced from dc. We denote by $(\gamma_v^{c,p}(t))$ the geodesic of (M, c) determined by an initial point $p \in M$ and an initial tangent vector v. We denote by $S_p^c(M)$ the unit tangent vectors v at p. Let $e \in S_p^c(M)$ be the unit tangent vector parallel to the positive x-axis. We can identify $v \in S_p^c(M)$ with the angle which v forms to e. The angular measure of $A \subset S_p^c(M)$ is denoted by $|A|_c$. E. Hopf classified (M, c) into the following two classes ([3], p. 271, Definition).

Definition (C). (M, c) is called of first class, if for some (or equivalently any) fixed $p \in M$,

$$|\{v \in S_p^c(M) ; \lim_{t\to\infty} d_c(\gamma_v^{c,p}(t), p) = \infty\}|_c = 0.$$

(M, c) is called of second class, if it is not of first class.

The following theorem is due to E. Hopf, M. Tsuji, etc. (see [3], p. 273, Hauptsatz 5.2 and p. 280, Hauptsatz 7.2).

Alternative theorem (C) ([5], [6]). *The following three conditions are equivalent*: (a) *(M, c) is of first class,* (b) *Γ is of divergence type,* (c) *the geodesic flow of (M, c) is ergodic.*

Moreover, the following three conditions are equivalent: (d) *(M, c) is of second class,* (e) *Γ is of convergence type,* (f) *the geodesic flow of (M, c) is dissipative.*

2. Theorem. Let f be a C^∞-function on D such that inequalities $0 < c_1 \leqq f(x, y) \leqq c_2$ hold on D, where c_1 and c_2 are constants. We give a Riemannian metric ds by

$$ds^2 = f^2(x, y)(dx^2 + dy^2) / (1 - x^2 - y^2)^2$$

on D. Let f be invariant under Γ. Then the Riemannian metric ds is naturally induced on M. We denote by (M, f) the 2-dimensional Riemannian manifold (M, ds) ([1], [2]). We denote by $(\gamma_v^p(t))$ the geodesic of (M, f) determined by an initial point p and an initial tangent vector v. We denote by $S_p(M)$ the unit tangent vectors v at p. The angular measure $|A|$ of $A \subset S_p(M)$ is introduced similarly to $|\cdot|_c$.

The following hypotheses are given in [3] (p. 294 and p. 300).

(H.A) The Gaussian curvature $K = K(x, y)$ of (M, f) satisfies the inequalities $-a_1 \leqq K(x, y) \leqq -a_2 < 0$ on M, where a_1 and a_2 are positive constants.

(H.B) The directional derivative dK/ds of K along any geodesic circle satisfies inequalities $-b \leqq dK/ds \leqq b$ on (M, f), where b is a constant independent of centers and radii of geodesic circles (see also [2], p. 234, (H3)).

Under (H.A) and (H.B), E. Hopf gave the following classification of (M, f), which is garanteed by Lemma 2 in §3 (see [3], p. 304).

Definition (V). Set $U^\infty(p) = \{v \in S_p(M)\,;\, \lim_{t\to\infty} d(\gamma_v^p(t), p) = \infty\}$. (M, f) is called of first class, if for some (or equivalently any) fixed $p \in M$, $|U^\infty(p)| = 0$. (M, c) is called of second class, if for some (or equivalently any) fixed $p \in M$, $|U^\infty(p)| > 0$.

The following Alternative theorem (V) is essential for us ([3], p. 304, Hauptsatz 15.2).

Alternative theorem (V). *If (M, f) is of first class, then the geodesic flow of (M, f) is ergodic. If (M, f) is of second class, then the geodesic flow of (M, f) is dissipative, that is, for any fixed $p \in M$ $|U^\infty(p)| = |S_p(M)|$.*

Our problem is to find a criterion of the classification of (M, f), when a fundamental domain of Γ is infinite area. The following theorem gives a partial answer to this problem.

Theorem. *Assume the following*:

(1) *A fundamental domain of Γ is of infinite area.*

(2) *The closure of the set $\{p \in M\,;\, f(p) \neq 2\}$ is compact.*

(3) *The Gaussian curvature K satisfies* (H.A) *and* (H.B).

Then the following three conditions are equivalent: (a) *(M, f) is of first class,* (b) *(M, c) is of first class,* (c) *Γ is of divergence type.*

Moreover, the following three are equivalent: (d) *(M, f) is of second class,* (e) *(M, c) is of second class,* (f) *Γ is of convergence type.*

From Alternative theorems (C) and (V), we obtain the following:

Corollary. *Under the assumptions in Theorem, the geodesic flow of (M, f) is ergodic, if and only if the geodesic flow of (M, c) is ergodic. Further, the geodesic flow of (M, f) is dissipative, if and only if the geodesic flow of (M, c) is dissipative.*

3. Lemma. To prove the theorem above we mention some preliminary facts. On $M = D/\Gamma$ we have two Riemannian metrics dc and ds, and two distances d_c and d induced from them respectively.

Lemma 1. *The inequalities*

$$(c_1/2)\, d_c(p, q) \leqq d(p, q) \leqq (c_2/2)\, d_c(p, q)$$

hold for any $p,\ q \in M$.

From the lemmas on the universal covering space (D, f) in [3], p. 302, Lemma 14.3, [4], p. 597, Hilfsatz 3.4 and [1], p. 211, 2, we have the following lemma, which is essential for a proof of our theorem. Fix a fundamental domain X of Γ and identify M to X. Then for $p,\ q \in X,\ v \in S_p(D)$ and $\theta \in \Gamma$, there exists uniquely $u \in S_{\theta q}(D)$ such that the geodesic lines $(\gamma_v^p(t))$ and $(\gamma_u^{\theta q}(t))$ in (D, f) are positively asymptotic, that is,

$$\lim_{t\to\infty} d(\gamma_v^p(t)),\ \gamma_u^{\theta q}(t+\alpha)) = 0$$

with a constant α independent of t. From this, a map $\Phi_{p,q}^{\theta}$ from $S_p(M)$ onto $S_q(M)$ can be defined by $\Phi_{p,q}^{\theta}(v) = u$ for each $\theta \in \Gamma$. For $p,\ a,\ q \in X$ and $\theta,\ \phi \in \Gamma$ we have

$$\text{(P)} \qquad \Phi_{a,q}^{\theta} \cdot \Phi_{p,a}^{\phi} = \Phi_{p,q}^{\phi\theta}, \quad (\Phi_{p,q}^{\theta})^{-1} = \Phi_{q,p}^{\theta^{-1}}.$$

Lemma 2. *We assume* (H.A) *and* (H.B) *for* K. *Let* $p,q \in M$ *be fixed. Then the map* $\Phi_{p,q}^{\theta}$ *from* $S_p(M)$ *to* $S_q(M)$ *satisfies the following properties*:
(1) *If* $\Phi_{p,q}^{\theta}(v) = u$, *then* $(\gamma_v^p(t))$ *and* $(\gamma_u^q(t))$ *are positively asymptotic in* $\boldsymbol{M}$.
(2) $\Phi_{p,q}^{\theta}$ *is* $1:1$, *onto, differentiable and its derivative* $d\Phi_{p,q}^{\theta}(v)/dv$ *is continuous and does not vanish for any* $v \in S_p(M)$.
(3) *We can find* $\theta = \theta(p,\ q) \in \Gamma$ *such that for* $\Phi_{p,q}^{\theta}(v) = u$ *and for any* $t \geqq 0$ *there exists* $\bar{t} \geqq 0$ *satisfying* $d(\gamma_v^p(\bar{t}),\ \gamma_u^q(t)) \leqq d(p,\ q)$.

4. Proof of theorem. Fix $p \in M$ and $r > 0$ such that the closed ball $B_r(p) = \{\bar{p} \in M\ ;\ d(p,\ \bar{p}) \leqq r\}$ includes the set $\{\bar{p} \in M\ ; f(\bar{p}) \neq 2\}$. Then at outside of $B_r(p)$, two Riemannian metrics ds and dc are coincident and $S_q^c(M) = S_q(M)$ for $q \notin B_r(p)$. Cover the set $\{\bar{p}\ ; d(p,\ \bar{p}) = 3r\}$ by $\cup_{q\in A} B_r(q)$, where A is a finite set with $d(p,\ q) = 3r,\ q \in A$. Put $W(q) = \{w \in S_q(M)\ ;\ \gamma_w^q(t) \notin B_r(p)$ for any $t \geqq 0$ and $\lim_{t\to\infty} d(q,\ \gamma_w^q(t)) = \infty\}$. For any $v \in U^\infty(p)$ there exists $1 = 1(v) > 0$ such that $d(p,\ \gamma_v^p(1)) = 3r$ and $d(p,\ \gamma_v^p(t)) \geqq 3r$ for all $t \geqq 1$. There exists $q \in A$ such that $a = \gamma_v^p(1) \in B_r(q)$. Put $h = [d\gamma_v^p(t)/dt]_{t=1}$. There exists $\theta = \theta(a,\ q) \in \Gamma$ such that $\Phi_{a,q}^{\theta}(h) = w$ satisfies (3) in Lemma 2. We have

$$\begin{aligned} d(p,\ \gamma_w^q(t)) &\geqq d(p,\ \gamma_v^p(1+\bar{t})) - d(\gamma_v^p(1+\bar{t}),\ \gamma_w^q(t)) \\ &\geqq 3r - d(\gamma_h^a(\bar{t}),\ \gamma_w^q(t)) \geqq 3r - r = 2r \end{aligned}$$

and

$$\begin{aligned} & d(q,\ \gamma_w^q(t)) \\ \geqq & -d(q,\ p) + d(p,\ \gamma_v^p(1+t+\alpha)) - d(\gamma_h^a(t+\alpha),\ \gamma_w^q(t)) \to \infty \end{aligned}$$

as $t \to \infty$, that is, $w \in W(q)$. Therefore from (P) $U^\infty(p) \subset \cup_{q\in A,\theta\in\Gamma} (\Phi_{p,q}^{\theta})^{-1} W(q) = \cup_{q\in A,\theta\in\Gamma} \Phi_{q,p}^{\theta}\, W(q)$. For $w \in W(q)$, the geodesic half-line $\{\gamma_w^q(t)\ ;\ t \geqq 0\}$ of (M, f) is in outside of $B_r(p)$ and hence it coinsides with the geodesic half-line $\{\gamma_w^{c,q}(t)\ ;\ t \geqq 0\}$ of (M, c). From Lemma 1, we have $\lim_{t\to\infty} d_c(q,\ \gamma_w^{c,q}(t)) = \infty$. Now we assume that (M, c) is of first class. Then $|W(q)| = |W(q)|_c = 0$ for $q \in A$. From Lemma 2, we have $|U^\infty(p)| \leqq \sum_{q\in A,\theta\in\Gamma} |\Phi_{q,p}^{\theta}\, W(q)| = 0$. This implies that (M, f) is of first class. Similarly we see, if (M, f) is of first class, then so is (M, c). By Alternative Theorem (C) we complete our proof.

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