24. Deformation Quantization of Poisson Algebras

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§ 0. **Introduction.** Let M be a C^∞ Poisson manifold, and $C^\infty(M)$ the set of all $\boldsymbol{C}$-valued C^∞ functions on M. In what follows, we put $\mathfrak{a}=C^\infty(M)$ for simplicity. By definition of Poisson manifolds, there exists a bilinear map $\{\ ,\ \}: \mathfrak{a}\times\mathfrak{a}\to\mathfrak{a}$, called the *Poisson bracket*, with the following properties: For any $f, g, h \in \mathfrak{a}$,

$$\begin{cases}\{f, g\}=-\{g, f\}, \quad \{f, g\cdot h\}=\{f, g\}\cdot h+g\cdot\{f, h\},\\ \{f, \{g, h\}\}+\{g, \{h, f\}\}+\{h, \{f, g\}\}=0.\end{cases}$$

The algebra $(\mathfrak{a}, \{\ ,\ \})$ is called the *Poisson algebra*.

We introduce the notion of the *deformation quantization* for the Poisson algebra $(\mathfrak{a}, \{\ ,\ \})$ as follows: Let $\mathfrak{a}[\![\nu]\!]$ be the direct product space $\prod_{m=0}^{\infty}\nu^m\mathfrak{a}$, where ν is a formal parameter. Consider an associative product $*: \mathfrak{a}[\![\nu]\!]\times\mathfrak{a}[\![\nu]\!]\to\mathfrak{a}[\![\nu]\!]$ such that ν is a center of $(\mathfrak{a}[\![\nu]\!], *)$ and 1 is the identity. Set for any $f, g \in \mathfrak{a}$, $f*g=\sum_{n=0}^{\infty}\nu^n\pi_n(f, g)$ according to the decomposition of $f*g$.

Let $\hat{\mathfrak{a}}_k=\mathfrak{a}[\![\nu]\!]/\nu^{k+1}\mathfrak{a}[\![\nu]\!]\equiv\mathfrak{a}\oplus\nu\mathfrak{a}\oplus\cdots\oplus\nu^k\mathfrak{a}$. Then an associative product $*$ can be considered on $\hat{\mathfrak{a}}_k$ by setting $\nu^{k+1}=0$. We denote this algebra by $(\hat{\mathfrak{a}}_k, \{\pi_m\}_{m=0}^k)$.

Definition. (i) For $k\geq 2$, an associative algebra $(\hat{\mathfrak{a}}_k, \{\pi_m\}_{m=0}^k)$ is called a *deformation quantization of order* k of $(\mathfrak{a}, \{\ ,\ \})$, if the following conditions are satisfied: For any $f, g \in \mathfrak{a}$.

(a) $\pi_0(f, g)=f\cdot g$ and $\pi_1(f, g)=-\frac{1}{2}\{f, g\}$.

(b) π_m is a bidifferential operator and $\pi_m(f, g)=(-1)^m\pi_m(g, f)$, $0\leq m\leq k$.

(ii) $(\mathfrak{a}[\![\nu]\!], *)$ is called a *deformation quantization* of $(\mathfrak{a}, \{\ ,\ \})$, if $(\hat{\mathfrak{a}}_k, \{\pi_m\}_{m=0}^k)$ is a deformation quantization of order k of $(\mathfrak{a}, \{\ ,\ \})$ for any k.

The main purpose of this paper is to study the obstructions for a deformation quantization $(\hat{\mathfrak{a}}_k, \{\pi_m\}_{m=0}^k)$ of order k to be extended to that of order $k+1$. The obstruction R_{k+1} is obtained as a deRham-Chevally 3-cocycle defined in the next section.

Our main theorem is stated as follows:

Main theorem. *Suppose* $(\hat{\mathfrak{a}}_k, \{\pi_m\}_{m=0}^k)$ *is a deformation quantization of order* k *of* $(\mathfrak{a}, \{\ ,\ \})$. *There exists a deformation quantization* $(\hat{\mathfrak{a}}_{k+1}, \{\pi_m\}_{m=0}^{k+1})$ *of order* $k+1$ *if and only if* $R_{k+1}=0$.

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§ 1. deRham-Chevally cohomology. For $\mathfrak{a}$, and $p\geq 1$, we denote by $C^p(\mathfrak{a})$ the sets of the continuous p-linear mappings from $\mathfrak{a}\times\cdots\times\mathfrak{a}$ to $\mathfrak{a}$, and denote by $AC^p(\mathfrak{a})$ the subset of $C^p(\mathfrak{a})$ with alternative properties. If $p=0$, we set $C^0(\mathfrak{a})=AC^0(\mathfrak{a})=\mathfrak{a}$.

For any $\pi\in C^2(\mathfrak{a})$, we define the *Hochschild coboundary operator* $\delta_\pi: C^p(\mathfrak{a})\to C^{p+1}(\mathfrak{a})$, $p\geq 1$ by

$$(\delta_\pi F)(v_1,\cdots,v_{p+1})=\pi(v_1,F(v_2,\cdots,v_{p+1}))+\sum_{i=1}^{p}(-1)^i F(v_1,\cdots,\pi(v_i,v_{i+1}),\cdots,v_{p+1})+(-1)^{p+1}\pi(F(v_1,\cdots,v_p),v_{p+1})$$

for $F\in C^p(\mathfrak{a})$. For $p=0$, we set $(\delta_\pi v)(v_1)=\pi(v_1,v)$ for any $v\in\mathfrak{a}$.

Given $\pi\in C^2(\mathfrak{a})$, we define $\partial_i^\pi: C^p(\mathfrak{a})\to C^{p+1}(\mathfrak{a})$ $i=1,\cdots,p$, $p\geq 1$ by

$$(1.1)\quad (\partial_i^\pi F)(v_1,\cdots,v_{p+1})=\pi(v_i,F(v_1,\cdots,\hat{v}_i,\cdots,v_{p+1}))-F(v_1,\cdots,\pi(v_i,v_{i+1}),\cdots,v_{p+1})+\pi(F(v_1,\cdots,\hat{v}_{i+1},\cdots,v_{p+1}),v_{i+1})$$

for any $F\in C^p(\mathfrak{a})$.

$F\in C^p(\mathfrak{a})$ is called a *p-derivation with respect to* π, if for any j, $(1\leq j\leq p)$

$$\partial_j^\pi F=0.$$

By $Der^p(\mathfrak{a},\pi)$, we denote the space of all p-derivations with respect to π. We also set

$$(1.2)\qquad \mathcal{A}^p(\mathfrak{a},\pi)=AC^p(\mathfrak{a})\cap Der^p(\mathfrak{a},\pi).$$

For any $\pi\in AC^2(\mathfrak{a})$, we define the *Chevalley coboudary operator* $d_\pi: AC^p(\mathfrak{a})\to AC^{p+1}(\mathfrak{a})$ by

$$(d_\pi F)(v_1,\cdots,v_{p+1})=\sum_{i=1}^{p+1}(-1)^{i+1}\pi(v_i,F(v_1,\cdots,\hat{v}_i,\cdots,v_{p+1}))+\sum_{i<j}(-1)^{i+j}F(\pi(v_i,v_j),v_1,\cdots,\hat{v}_i,\cdots,\hat{v}_j,\cdots,v_{p+1}).$$

Using the above coboundary operators, we will give a cohomology group for the Poisson algebra $(\mathfrak{a},\{\ ,\ \})$. In the following, we set $\pi_0(f,g)=f\cdot g$ and $\pi_1(f,g)=-\frac{1}{2}\{f,g\}$ for any $f,g\in\mathfrak{a}$. We easily have

Lemma 1.1. $d_{\pi_1}\mathcal{A}^p(\mathfrak{a},\pi_0)\subset\mathcal{A}^{p+1}(\mathfrak{a},\pi_0)$, $d_{\pi_1}^2=0$.

Thus, we can give the following p-th cohomology group $H^p(\mathfrak{a},\{\ ,\ \})$ of the cochain complex:

$$\cdots\longrightarrow\mathcal{A}^p(\mathfrak{a},\pi_0)\xrightarrow{d_{\pi_1}}\mathcal{A}^{p+1}(\mathfrak{a},\pi_0)\longrightarrow\cdots,$$

which is called the *deRham-Chevally cohomology group* of the Poisson algebra.

Remark. This cohomology group coincides the pure 1-differentiable cohomology group defined by Lichnerowicz [2]. If M is a symplectic manifold, then $H^p(M,\{\ ,\ \})$ is isomorphic to the usual p-th deRham cohomology group.

§ 2. Proof of main theorem. We give a brief sketch of proof of the main theorem. For the Poisson algebra $(\mathfrak{a},\{\ ,\ \})$ and any integer $k\geq 2$, suppose $\pi_2,\cdots,\pi_k$ are given as elements of $C^2(\mathfrak{a})$. π_m can be considered naturally as a bilinear map on $\mathfrak{a}[\![\nu]\!]$. We consider the following quantities which has been stated in the main theorem:

Definition 2.1. For each integer m, $1\leq m\leq k+1$, we set

(2.1) $$R_m=\frac{1}{2}\sum_{i+j=m,i,j\geq 1} d_{\pi_i^-}\pi_j^-,$$

where $\pi_i^-(f,g)=\frac{1}{2}(\pi_i(f,g)-\pi_i(g,f))$.

Note that R_m vanishes automatically for odd m. In the following, we assume that $(\hat{\mathfrak{a}}_k,\{\pi_m\}_{m=0}^k)$ is a deformation quantization of order k of $(\mathfrak{a},\{\ ,\ \})$.

Lemma 2.1. *R_{k+1} is a deRham-Chevally 3-cocycle.*

The Jacobi identity for the algebra $(\hat{\mathfrak{a}}_k,\{\pi_m\}_{m=0}^k)$ yields the following:

Lemma 2.2. *If $(\hat{\mathfrak{a}}_k,\{\pi_m\}_{m=0}^k)$ is a deformation quantization of order k of $(\mathfrak{a},\{\ ,\ \})$, then $R_m=0$ for $2\leq m\leq k$.*

By Lemma 2.2, if there exists a deformation quantization $(\hat{\mathfrak{a}}_{k+1},\{\pi_m\}_{m=0}^{k+1})$ of order $k+1$, then $R_{k+1}=0$. Notice that R_{k+1} is expressed only by using $\pi_1,\cdots,\pi_k$.

Suppose now $R_{k+1}=0$. We have only to solve the equation

(2.2) $$\delta_{\pi_0}\pi_{k+1}=-\frac{1}{2}\sum_{i+j=k+1,i,j\geq 1}\delta_{\pi_i}\pi_j$$

with the additional condition (b).

To solve (2.2), we work in each local coordinate neighborhood $(U;x_1,\cdots,x_n)$. We construct $\pi_{k+1}(x^\alpha,x^\beta)$ inductively with respect to the degree $|\alpha|+|\beta|$ by solving huge over determined linear systems. Namely, the construction is divided into odd and even cases for $k+1$. For even $k+1$, the solubility condition of (2.2) with (b) is given by $R_{k+1}=0$. On the other hand, for odd $k+1$, we can solve $\pi_{k+1}\in AC^2(\mathfrak{a})$ with (2.2) directly by using $R_m=0$ $(0\leq m\leq k+1)$. For making $\pi_{k+1}(x^\alpha,x^\beta)$, we need a long direct computation. It looks like impossible to get π_{k+1} only by the cohomological computations.

§ 3. Other results and examples. As easy consequences of the main theorem, we have the following results on the deformation quantizability of Poisson manifolds.

In the case of dim $M=2$, $R_m=0$ trivially for any m. Thus, following the proof of Main theorem, we have

Corollary 3.1. *For any Poisson 2-manifold, there exists a deformation quantization of $(\mathfrak{a},\{\ ,\ \})$.*

Similarly, we have

Corollary 3.2. *Let M be a Poisson manifold. Assume $H^3(M,\{\ ,\ \})=0$. Then, for any cohomology class $[\theta]\in H^2(M,\{\ ,\ \})$, there exists a deformation quantization $(\mathfrak{a}[\![\nu]\!],*_{[\theta]})$. Moreover, if for given two cohomology classes $[\theta]$, $[\theta']\in H^2(M,\{\ ,\ \})$, there exists an isomorphism*

$$\phi:(\mathfrak{a}[\![\nu]\!],*_{[\theta]})\cong(\mathfrak{a}[\![\nu]\!],*_{[\theta']})$$

such that $\phi=1\ (\mathrm{mod}\ \nu^2)$ and $\phi(\nu)=\nu$, then $[\theta]=[\theta']$.

If the first obstruction cocycle R_4 is not a coboundary, then $(\mathfrak{a},\{\ ,\ \})$ has no deformation quantization. R_4 relates to the anomaly in the Jacobi identity of [6]. However, we do not know whether there is a Poisson algebra with $R_4\not\sim 0$.

Since any symplectic manifold is deformation quantizable [1], [4], we see that the condition $H^3(M, \{\ ,\ \})=0$ is not a necessary condition for a Poisson algebra to be deformation quantizable. Quantizability seems to relate to local structures of singularities of Poisson structure where the rank is changing.

The following examples were found by means of our proof of Main theorem:

Ex. 1. Let x, y, z be the natural coordinate functions on $\boldsymbol{R}^3$. For any positive integers k, l, m, the relations $\{x, y\}=z^k$, $\{y, z\}=x^l$, $\{z, x\}=y^m$ define a Poisson algebra structure on $C^\infty(\boldsymbol{R}^3)$. The Poisson algebra $(C^\infty(\boldsymbol{R}^3), \{\ ,\ \})$ has a deformation quantization such that $\pi_j(x, y)=\pi_j(y, z)=\pi_j(z, x)=0$ for $j\geq 2$. The obtained deformed algebra is characterized by the relations $[x, y]=-\nu z^k$, $[y, z]=-\nu x^l$, $[z, x]=-\nu y^m$ where $z^k=(z\cdot)^k=(z*)^k$ etc.

Ex. 2. Let $x_1, x_2, \cdots, x_n$ be the natural coordinate functions on $\boldsymbol{R}^n$. For any skew-symmetric matrix $(a_{ij})_{1\leq i,j\leq n}$, the relations $\{x_i, x_j\}=a_{ij}x_i x_j$, $(1\leq i,\ j\leq n)$ define a Poisson algebra structure on $C^\infty(\boldsymbol{R}^n)$. Then, $(C^\infty(\boldsymbol{R}^n)$, $\{\ ,\ \})$ has a deformation quantization. This relates to a non-commutative torus (cf. [5]).

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