73. Dimension Estimate of the Global Attractor for Resonant Motion of a Spherical Pendulum

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1. Introduction and result. In [8] Miles derived the following system (SP). It describes the motion of a lightly damped spherical pendulum, which is forced to oscillate horizontally in the neighborhood of resonance:

$$\text{(SP)} \qquad \begin{cases} \dfrac{dp_1}{dt} = -\alpha p_1 - \left(\nu + \dfrac{E}{8}\right) q_1 - \dfrac{3}{4} M p_2, \\ \dfrac{dq_1}{dt} = -\alpha q_1 + \left(\nu + \dfrac{E}{8}\right) p_1 - \dfrac{3}{4} M q_2 + 1, \\ \dfrac{dp_2}{dt} = -\alpha p_2 - \left(\nu + \dfrac{E}{8}\right) q_2 + \dfrac{3}{4} M p_1, \\ \dfrac{dq_2}{dt} = -\alpha q_2 + \left(\nu + \dfrac{E}{8}\right) p_2 + \dfrac{3}{4} M q_1, \end{cases}$$

where $\alpha > 0$ and $\nu \in \boldsymbol{R}$ represent a damping coefficient and a frequency offset, respectively. Here $(p_1(t), q_1(t), p_2(t), q_2(t))$ denotes slowly varying amplitudes of degenerate modes 1 and 2 in a four dimensional phase space, and we have set $E = E(t) := p_1(t)^2 + q_1(t)^2 + p_2(t)^2 + q_2(t)^2$, $M = M(t) := p_1(t)q_2(t) - p_2(t)q_1(t)$.

The aim of this paper is to estimate an upper bound for the dimension of X analytically. Basically we make use of the Kaplan-Yorke formula. This formula connects the upper bound with the Lyapunov exponents. This was conjectured by Kaplan and Yorke [7] and proved by Constantin and Foias [1]. In Eden, Foias and Temam [4], this enables to estimate the dimension of a global attractor for the Lorenz system. (SP) consists of four equations unlike the Lorenz system. We therefore adopt the technique used in Ishimura and Nakamura [6].

Now we state our main result.

Theorem. *Let X be the maximal compact invariant set of* (SP). *Let* $\dim_{\mathscr{H}}$ *denote the Hausdorff dimension. For any $\nu \in \boldsymbol{R}$, we have the following:*

(i) *If $0 < \alpha^3 \leq \frac{1}{3}$, then*

$$\dim_{\mathscr{H}}(X) \leq 3 + \frac{-3\alpha^3 + 1}{\alpha^3 + 1}.$$

(ii) *If $\frac{1}{3} < \alpha^3 \leq \frac{9}{16}$, then*

$$\dim_{\mathscr{H}}(X) \leq 2 + \frac{-16\alpha^3 + 9}{8\alpha^3 + 1}.$$

(iii) *If* $\frac{9}{16} < \alpha^3 \leq 1$, *then*

$$\dim_{\mathscr{H}}(X) \leq 1 + \frac{-8\alpha^3 + 8}{8\alpha^3 - 1}.$$

(iv) *If* $\alpha^3 > 1$, *then* X *is a linearly stable invariant set.*

Remark that Lemma 2.4 enables us to obtain this above X in a general theory of dynamical systems. (For example we refer to Temam [11].) In a forthcoming paper [5], we shall study a more general system analytically and numerically.

2. Sketch of the proof. We first recall some notations and known results concerning the Lyapunov exponent and the Hausdorff dimension. For the proof and other properties, we refer to texts of Eden, Foias and Temam [4], Constantin and Foias [2] and Ladyzhenskaya [8].

Let $\{S(t)\}_{t\geq 0}$ be a C^0-semigroup of injective operators acting on a separable Hilbert space H. We assume that there exists a compact set X such that $S(t)X = X$ for all $t \geq 0$. For all $u_0 \in X$ we assume that there exists a compact linear oprator $S'(t, u_0)$ on H satisfying

$$\| S(t)u_1 - S(t)u_0 - S'(t, u_0)(u_1 - u_0) \| \leq C(t)o(\| u_1 - u_0 \|),$$

for some nondecreasing function $C(t)$.

We define $\mu_i(u_0)$'s and μ_i's as follows:

$$(\mu_1 + \mu_2 + \cdots + \mu_N)(u_0) := \limsup_{t\to\infty} \frac{1}{t} \log \sup_{\| \wedge_{i=1}^N v_{0i} \| \leq 1} \| \bigwedge_{i=1}^N S'(t, u_0)v_{0i} \|,$$

$$\mu_1 + \mu_2 + \cdots + \mu_N := \limsup_{t\to\infty} \sup_{u_0\in X} \frac{1}{t} \log \sup_{\| \wedge_{i=1}^N v_{0i} \| \leq 1} \| \bigwedge_{i=1}^N S'(t, u_0)v_{0i} \|.$$

Here $\wedge$ means the exterior product. Remark that μ_i's are called global Lyapunov exponents and $\mu_i(u_0)$'s local Lyapunov exponents.

We next recall the definition of the Hausdorff dimension: Let X be a compact subset of H. We set

$$\mu_{d,\varepsilon}(X) := \inf \left\{ \sum_{i=1}^k r_i^d \,;\, r_i \leq \varepsilon,\ X \subseteq \cup_{i=1}^k B_{r_i},\ k \in \boldsymbol{Z} \right\}.$$

Here B_{r_i} denotes the ball with radius r_i. $\mu_{d,\varepsilon}(X)$ is a nonincreasing function of ε. So we can define

$$\mu_d(X) := \sup_{\varepsilon>0} \mu_{d,\varepsilon}(X) = \lim_{\varepsilon\searrow 0} \mu_{d,\varepsilon}(X).$$

The Hausdorff dimension $\dim\mathscr{H}$ is then given by

$$\dim_{\mathscr{H}} := \inf\{d > 0\,;\, \mu_d(X) = 0\}.$$

Now we present the Kaplan-Yorke formura, which will be the main ingredient of the proof.

Theorem 2.1 (Kaplan-Yorke formula). *Let* N *be the first integer such that*

$$(\mu_1 + \mu_2 + \cdots + \mu_N + \mu_{N+1})(u_0) < 0$$

for all $u_0 \in X$. *Then we have*

$$\dim_{\mathscr{H}}(X) \leq \sup_{u_0\in X} \left\{ N + \frac{(\mu_1 + \mu_2 + \cdots + \mu_N)(u_0)}{|\mu_{N+1}(u_0)|} \right\}.$$

Corollary 2.2 (Constantin, Foias and Temam [3; Theorem 3.3]). *Suppose*

$\mu_1 + \mu_2 + \cdots + \mu_N + \mu_{N+1} < 0$, *Then we have*

$$\dim_{\mathscr{H}}(X) \leq N + \frac{\mu_1 + \mu_2 + \cdots + \mu_N}{|\mu_{N+1}|}.$$

We next state several lemmas needed for a proof of our main theorem. At $u_0 = {}^t(p_1, q_1, p_2, q_2) \in \boldsymbol{R}^4$, the matrix L for the linearized system of (SP) is given by

$$L = (L_{ij}),$$

where

$$L_{11} = -\alpha - \frac{1}{4}p_1q_1 - \frac{3}{4}p_2q_2, \qquad L_{12} = -\left(\nu + \frac{E}{8}\right) - \frac{1}{4}q_1^2 + \frac{3}{4}p_2^2,$$

$$L_{13} = \frac{5}{4}p_2q_1 - \frac{3}{4}p_1q_2, \qquad L_{14} = -\frac{1}{4}q_1q_2 - \frac{3}{4}p_1p_2,$$

$$L_{21} = \left(\nu + \frac{E}{8}\right) + \frac{1}{4}p_1^2 - \frac{3}{4}q_2^2, \qquad L_{22} = -\alpha + \frac{1}{4}p_1q_1 + \frac{3}{4}p_2q_2,$$

$$L_{23} = \frac{1}{4}p_1p_2 - \frac{3}{4}q_1q_2, \qquad L_{24} = -\frac{5}{4}p_1q_2 + \frac{3}{4}p_2q_1,$$

$$L_{31} = \frac{5}{4}p_1q_2 - \frac{3}{4}p_2q_1, \qquad L_{32} = -\frac{1}{4}q_1q_2 - \frac{3}{4}p_1p_2,$$

$$L_{33} = -\alpha - \frac{1}{4}p_2q_2 - \frac{3}{4}p_1q_1, \qquad L_{34} = -\left(\nu + \frac{E}{8}\right) - \frac{1}{4}q_2^2 + \frac{3}{4}p_1^2,$$

$$L_{41} = \frac{1}{4}p_1p_2 + \frac{3}{4}q_1q_2, \qquad L_{42} = -\frac{5}{4}p_2q_1 + \frac{3}{4}p_1q_2,$$

$$L_{43} = \left(\nu + \frac{E}{8}\right) + \frac{1}{4}p_2^2 - \frac{3}{4}q_1^2, \qquad L_{44} = -\alpha + \frac{1}{4}p_2q_2 + \frac{3}{4}p_1q_1.$$

Let $S(t)$ denote the solution operator for (SP); i.e. $S(t)u_0 = u(t)$ for the initial value $u_0 \in \boldsymbol{R}^4$. We consider the solution $v_i(t) = S'(t, u_0)v_{0_i}$ $(i = 1,2,3,4)$ of the eqation

$$\text{(2.1)} \qquad \begin{cases} \dfrac{dv_i}{dt} = Lv_i, \\ v_i(0) = v_{0i}. \end{cases}$$

Invoking Lemma 3.5 in Constantin and Foias [2] for our situation, we get the following equation:

$$\text{(2.2)} \qquad \frac{d}{dt}|v_1 \wedge v_2 \wedge v_3 \wedge v_4|^2 = 2(trL)|v_1 \wedge v_2 \wedge v_3 \wedge v_4|^2.$$

From (2.2) and the definition of the Lyapunov exponent, we have

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = -4\alpha < 0.$$

Let $\{e_i\}_{i=1}^4$ denote the standard basis for $\boldsymbol{R}^4$. And we set

$$\text{(2.3)} \qquad v_i(t) = \sum_{j=1}^4 v_i^j(t)e_j \quad (i = 1,2,3,4).$$

Then we have the following.

Lemma 2.3. *Suppose that each v_i solves the equation* (2.1), *then we have*

$$\text{(2.4)} \qquad \frac{d}{dt}|v_1 \wedge v_2 \wedge v_3|^2 \leq 2(-3\alpha + E)|v_1 \wedge v_2 \wedge v_3|^2,$$

$$\text{(2.5)} \qquad \frac{d}{dt}|v_1 \wedge v_2|^2 \leq 2\left(-2\alpha + \frac{9}{8}E\right)|v_1 \wedge v_2|^2,$$

(2.6) $$\frac{d}{dt}|v_1|^2 \leq 2(-\alpha + E)|v_1|^2.$$

Now we shall state the asymptotic behavior of a solution of (SP). Any solution of (SP) has the following property:

Lemma 2.4. *Let* $(p_1(t), q_1(t), p_2(t), q_2(t))$ *denote any solution of* (SP). *Then for any* $\varepsilon > 0$, *there exists* $t_0 = t_0(\varepsilon) > 0$ *such that for any* $t > t_0$

$$|E(t)| < \frac{1}{\alpha^2} + \varepsilon.$$

Especially suppose that $(p_1(t), q_1(t), p_2(t), q_2(t)) \in X$, *then we have*

$$|E(t)| \leq \frac{1}{\alpha^2}, \text{ for any } t \geq 0.$$

Here X *is a compact global attractor.*

Now we can prove our main theorem. It follows from Lemmas 2.3, 2.4 and the definition of the Lyapunov exponent that for each Lyapunov exponent μ_i of X, we have the following:

(2.7) $$\mu_1 + \mu_2 + \mu_3 \leq -3\alpha + \frac{1}{\alpha^2},$$

(2.8) $$\mu_1 + \mu_2 \leq -2\alpha + \frac{9}{8\alpha^2},$$

(2.9) $$\mu_1 \leq -\alpha + \frac{1}{\alpha^2}.$$

By Corollary 2.2 and (2.7)-(2.9), we obtain our theorem. Indeed, suppose $\mu_1 + \mu_2 + \mu_3 \leq M$ for some $M \in \boldsymbol{R}$. When $M \geq 0$, then $|\mu_4| = -\mu_4 = \mu_1 + \mu_2 + \mu_3 + 4\alpha$. Invoking Kaplan-Yorke formula, we have

$$\dim_{\mathscr{H}}(X) \leq 3 + \frac{\mu_1 + \mu_2 + \mu_3}{|\mu_4|} \leq 3 + \frac{M}{4\alpha + M}.$$

When $M < 0$, we have $\mu_1 + \mu_2 + \mu_3 \leq M < 0$. Then we can go to the next step. Here suppose $\mu_1 + \mu_2 \leq N$ for some $N \in \boldsymbol{R}$. When $N \geq 0$,

$$\dim_{\mathscr{H}}(X) \leq 2 + \frac{\mu_1 + \mu_2}{|\mu_3|} \leq 2 + \frac{N}{-M + N}.$$

When $N < 0$, we have $\mu_1 + \mu_2 \leq N < 0$. Then we can go ahead again. Here suppose $\mu_1 \leq K$ for some $K \in \boldsymbol{R}$. When $K \geq 0$,

$$\dim_{\mathscr{H}}(X) \leq 1 + \frac{K}{-N + K}.$$

When $K < 0$, we have $\mu_1 < 0$. X is therefore linearly stable.

By (2.7)-(2.9) we can choose $M := -3\alpha + \frac{1}{\alpha^2}$, $N := -2\alpha + \frac{9}{8\alpha^2}$, and $K := -\alpha + \frac{1}{\alpha^2}$ to obtain our theorem.

References

[1] P. Constantin and C. Foias: Global Lyapunov exponents, Kaplan-Yorke formulas and the dimension of the attractor for $2D$ Navier-Stokes equations. Comm. Pure. Appl. Math., **38**, 1–27 (1985).

[2] ——: Navier-Stokes equations. Chicago Lectures in Math, Chicago (1988).

[3] P. Constantin, C. Foias and R. Temam: Attractors representing turbulent flows.

Amer. Math. Soc. Memoirs, vol. 53, no. 314 (1985).

[4] A. Eden, C. Foias and R. Temam: Local and global Pyaounov exponents. J. Dyn. Diff. Eqs., **3**, 133–177 (1991).

[5] Y. Hattori, *et al.*: Dimension estimate of the global attractor for forced oscillation systems (preprint).

[6] N. Ishimura and M. Nakamura: Estimate on the Hausdorff dimension of the global attractor for the simplified magnetic Bénard system (preprint).

[7] J. Kaplan and J. Yorke: Chaotic behaviour of multidimensional difference equations. Lect. Notes in Math., vol.730, 204–227 (1979).

[8] O. Ladyzhenskaya: Attractors for Semigroups and Evolution Equations. Leziont Lincee, Cambridge Univ. Press, Cambridge (1991).

[9] J. W. Miles: Stability of forced oscillations of a spherical pendulum. Quart. Appl. Math., **20**, 21–32 (1962).

[10] ——: Resonant motion of a spherical pendulum. Phisica, **11D**, 309–323 (1984).

[11] R. Temam: Infinite Dimensional Dynamical Systems in Mechanics and Physics. Springer-Verlag, Berlin (1988).