67. *Algebraic Tori Admitting Finite Central Coregular Extensions*

By Haruhisa NAKAJIMA*)

Department of Mathematics, Keio University

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Abstract: We determine representations of algebraic complex tori admitting finite central coregular extensions.

Key words: Algebraic torus; coregular representation; affine semigroup ring.

1. Introduction. Let $\rho : G \to GL(V(\rho))$ be a finite dimensional rational representation of a complex reductive algebraic group G over the field $\boldsymbol{C}$ of complex numbers, where $V(\rho)$ denotes the representation space. A representation $\varphi : H \to GL(V(\varphi))$ of an algebraic group H is said to be a *finite extension* of ρ or of (ρ, G), if $V(\varphi) = V(\rho)$ and there is a morphism $\psi : G \to H$ such that $\rho = \varphi \circ \psi$ and the index of the canonical image of G in H is finite. Moreover if (φ, H) is coregular, i.e., if its associated quotient variety $V(\varphi)/H = \mathrm{Spec}(\boldsymbol{C}[\varphi]^H)$ is an affine space, then (φ, H) or H is said to be a *finite coregular extension* of ρ and we also say that ρ admits a finite coregular extension, where $\boldsymbol{C}[\varphi]$ denotes the affine coordinate ring of $V(\varphi)$. A finite extension (φ, H) of ρ is said to be *central*, if H is generated by the union of G and the centralizer $Z_H(G)$ of G in H. According to [7], in 1991, D. Shmel'kin has classified all finite coregular extensions of irreducible representations of connected complex simple algebraic groups. Recently, in [7], D. I. Panyushev has defined finite coregular extensions and showed that the associated quotient varieties of the representations of connected semisimple algebraic groups admitting finite coregular extensions are complete intersections. This implies that D. Shmel'kin's classification is *a priori* related to the author's one in [5] (cf.[7]).

Hereafter G stands for a connected complex algebraic torus. Simplicial torus embeddings are defined in [3]. The purpose of this paper is to show

Theorem 1.1. *(ρ, G) admits a finite central coregular extension if and only if the rational convex polyhedral cone associated with the torus embedding $V(\rho)/G$ is simplicial.*

As an easy consequence of this theorem, we obtain the following criterion: Let $\{Y_1, \ldots, Y_n\}$ be a basis of $V(\rho)$ on which $\rho(G)$ is a diagonal subgroup of $GL_n(\boldsymbol{C})$. Let Γ denote a set consisting of all minimal subsets Λ of $\{1, \ldots, n\}$ such that nonzero weights in each subspace $\sum_{i \in \Lambda} \boldsymbol{C} Y_i$ generate a

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positive half plane containing the origin.

Corollary 1.2. *(ρ, G) admits a finite central coregular extension if and only if* $\bigcup_{\substack{\Theta\in\Gamma \\ \Theta\neq\Lambda}} \Theta \not\supseteq \Lambda$ *for all* $\Lambda \in \Gamma$. Q.E.D.

In [6] we have determined finite (but may not be central) coregular extensions of a connected algebraic torus of rank one and, as an application, have announced a refinement of Theorem 2 of [2]. There exist representations of algebraic tori which admit finite coregular extensions and do not admit finite central ones. In the preparation of the proof, the positively graded algebras whose Segre products are polynomial rings are determined. Our method shall be used also in a forthcoming paper of the author.

2. The Segre products. Let $A^i = \bigoplus_{j=0}^{\infty} A^i_j$, $i = 1,2$, be noetherian positively graded algebras defined over $A^1_0 = A^2_0 = \boldsymbol{C}$ of dimension ≥ 1. We denote by $A^1 \#_{\boldsymbol{C}} A^2$ the Segre product $\bigoplus_{j=0}^{\infty} (A^1_j \otimes_{\boldsymbol{C}} A^2_j)$ of graded algebras A^i, $i = 1,2$. The multiplicative group $\boldsymbol{C}^*$ acts on A^i, $i = 1,2$, respectively as $\boldsymbol{C}$-algebra automorphisms in such a way that, for $j \geq 0$, each element in A^1_j (resp. A^2_j) is of weight $\nu(j)$(resp. $\nu(-j)$), where ν denotes an isomorphism $\boldsymbol{Z} \to \mathrm{Hom}(\boldsymbol{C}^*, \boldsymbol{C}^*)$. Then $A^1 \#_{\boldsymbol{C}} A^2 = (A^1 \otimes A^2)^{\boldsymbol{C}^*}$.

Lemma 2.1. *Let f_i be a homogeneous element of A^i $(i = 1,2)$. If $f_1 \otimes f_2$ is $A^1 \#_{\boldsymbol{C}} A^2$-regular, then* $\mathrm{ht}\, A^i f_i = 1$.

Proof. We express $\mathrm{nil}\, A^1 = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_m \cap \cdots \cap \mathfrak{P}_n$ for distinct minimal prime divisors $\mathfrak{P}_i$ of $\{0\}$ in A^1. We assume that $f_1 \in \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_m$ and $f_1 \notin \mathfrak{P}_i$ $(m < i \leq n)$. Choose a homogeneous element h from $\mathfrak{P}_{m+1} \cap \cdots \cap \mathfrak{P}_n$ such that $h \notin \mathfrak{P}_j (1 \leq j \leq m)$. Let a, b be natural numbers which satisfy $h^a \otimes f_2^b \in A^1 \#_{\boldsymbol{C}} A^2$. Then $(h^a \otimes f_2^b) \cdot (f_1 \otimes f_2) \in \mathrm{nil}(A^1 \#_{\boldsymbol{C}} A^2)$. From this we immediately derive a contradiction. Q.E.D.

Lemma 2.2*⁾**.** *If $A^1 \#_{\boldsymbol{C}} A^2$ is a polynomial ring over $\boldsymbol{C}$, then one of $A^i (i = 1,2)$ is of dimension one.*

Proof. Deny the assertion. We can choose homogeneous elements $f_i \in A^i$ such that $f_1 \otimes f_2$ is a member of a regular system of homogeneous parameters of $A^1 \#_{\boldsymbol{C}} A^2$. Since the functor $\cdot \#_{\boldsymbol{C}} A^2$ is exact, we have $A^1 \#_{\boldsymbol{C}} A^2 / (A^1 f_1 \#_{\boldsymbol{C}} A^2) \cong (A^1 / A^1 f_1) \#_{\boldsymbol{C}} A^2$. Using Theorem 4.2.3 of [1] and Lemma 2.1, we see that $(A^1/A^1 f_1) \#_{\boldsymbol{C}} A^2$ is a polynomial ring over $\boldsymbol{C}$ of dimension $\dim A^1 + \dim A^2 - 2$ and $A^1 f_1 \#_{\boldsymbol{C}} A^2 = (f_1 \otimes f_2) \cdot (A^1 \#_{\boldsymbol{C}} A^2)$. By induction on dimension, we need to treat only the case where $\dim A^1 = \dim A^2 = 2$. Then $A^1 \#_{\boldsymbol{C}} A^2 = \boldsymbol{C}[x_1 \otimes y_1, x_2 \otimes y_2, x_3 \otimes y_3]$ for some homogeneous $x_i \in A^1$ and $y_i \in A^2$. By the observation stated as above, we see $A^1 x_1 \#_{\boldsymbol{C}} A^2 = (x_1 \otimes y_1) \cdot (A^1 \#_{\boldsymbol{C}} A^2)$ and $(A^1/A^1 x_1) \#_{\boldsymbol{C}} A^2 \cong \boldsymbol{C}[\overline{x_2 \otimes y_2}, \overline{x_3 \otimes y_3}]$, where $\overline{x_i \otimes y_i}$ is the canonical image of $x_i \otimes y_i$ in $(A^1 \#_{\boldsymbol{C}} A^2)/(A^1 x_1 \#_{\boldsymbol{C}} A^2)$. Since the canonical image of $(x_2 \otimes y_2, x_3 \otimes y_3)$ forms a regular system of homogeneous parameters of $A^1 \#_{\boldsymbol{C}} (A^2/A^2 y_1)$, (y_1, y_2) is a system of homogeneous parameters of A^2(cf. Lemma 2.1). We can choose natural numbers d, e such that $x_1^d \otimes y_2^e = w \cdot (x_1 \otimes y_1)$ for some $w \in A^1 \#_{\boldsymbol{C}} A^2$. Because x_1 is not nilpotent, we get a specialization $\mu : A^1 \to \boldsymbol{C}$ satisfying $\mu(x_1) \neq 0$ and apply $\mu \otimes 1$ to $A^1 \otimes_{\boldsymbol{C}} A^2$.

*⁾ Although this was shown in [6], we give the proof for reader's convenience.

Consequently we see that $y_2^e \in y_1A^2$, which is a contradiction. Q.E.D.

Furthermore, we suppose that one of graded algebras in the set $\{A^i \mid \dim A^i = 1\}$ is a normal domain, unless it is empty. For each i, let e_i denote the largest common divisor of integers in the unique minimal set of generators of the additive subsemigroup of $\boldsymbol{Z}$ generated by $\{j \in \boldsymbol{Z} \mid A_j^i \neq \{0\}\}$.

Proposition 2.3. *$A^1 \#_{\boldsymbol{C}} A^2$ is a polynomial ring over $\boldsymbol{C}$ if and only if both graded subalgebras $\bigoplus_{j \in \boldsymbol{Z}e_2} A_j^1$, $\bigoplus_{j \in \boldsymbol{Z}e_1} A_j^2$, are polynomial rings over $\boldsymbol{C}$ and one of $A^i (i = 1,2)$ is of dimension one.*

Proof. By Lemma 2.2 we may suppose that A^1 is a graded polynomial ring $\boldsymbol{C}[X]$ of dimension one. Using the action of $\boldsymbol{C}^*$, we see that $(A^1)^{\mathrm{Ker}(\boldsymbol{C}^* \to \mathrm{Aut}A^2)} = \bigoplus_{j \in \boldsymbol{Z}e_2} A_j^1$ and $(A^2)^{\mathrm{Ker}(\boldsymbol{C}^* \to \mathrm{Aut}A^1)} = \bigoplus_{j \in \boldsymbol{Z}e_1} A_j^2$, which imply $A^1 \#_{\boldsymbol{C}} A^2 \cong (\bigoplus_{j \in \boldsymbol{Z}e_2} A_j^1) \#_{\boldsymbol{C}} (\bigoplus_{j \in \boldsymbol{Z}e_1} A_j^2)$ and $\bigoplus_{j \in \boldsymbol{Z}e_2} A_j^1$ is a graded polynomial ring. Let $\{f_1, \ldots, f_m\}$ be a minimal system of homogeneous generators of $\bigoplus_{j \in \boldsymbol{Z}e_1} A_j^2$ as a $\boldsymbol{C}$-algebra. Then $\{X^{\deg f_1/e_1} \otimes f_1, \ldots, X^{\deg f_m/e_1} \otimes f_m\}$ is a minimal system of homogeneous generators of $A^1 \#_{\boldsymbol{C}} A^2 \cong A^1 \#_{\boldsymbol{C}} (\bigoplus_{j \in \boldsymbol{Z}e_1} A_j^2)$. Thus the assertion follows from this, because $\dim A^1 \#_{\boldsymbol{C}} A^2 = \dim A^2$. Q.E.D.

3. Graded automorphisms of $\boldsymbol{C}[\rho]^G$. We regard any matrix in $GL_n(\boldsymbol{C})$ as an automorphism of $V(\rho)$ through the basis $\{Y_1, \ldots, Y_n\}$ defined in Sect. 1 and regard $\boldsymbol{C}[\rho]$ and $\boldsymbol{C}[\rho]^G$ as $\boldsymbol{Z}^n$-graded $\boldsymbol{C}$-algebras by the basis $\{X_1, \ldots, X_n\}$ of $V(\rho)^*$ dual to that basis. We may assume that $\{i \mid \boldsymbol{C}[\rho]X_i \cap \boldsymbol{C}[\rho]^G \neq \{0\}\} = \{1, \ldots, n'\}$. Let U be the subspace of $V(\rho)$ generated by $\{Y_1, \ldots, Y_{n'}\}$.

Lemma 3.1. *Let σ be a $\boldsymbol{C}$-algebra automorphism of $\boldsymbol{C}[\rho]^G$ preserving its $\boldsymbol{Z}^n$-gradation. Then there is a matrix $\hat{\sigma}$ in $GL(U)$ which induces σ and is diagonal on $\{Y_1, \ldots, Y_{n'}\}$.*

Proof. Let a set $\{M_1, \ldots, M_l\}$ of monomials of $\{X_1, \ldots, X_n\}$ be a minimal system of $\boldsymbol{Z}^n$-homogeneous generators of $\boldsymbol{C}[\rho]^G$. Put $\mathfrak{M} = \sum_{i=1}^{l} \boldsymbol{C}[\rho]^G (M_i - \sigma(M_i)/M_i)$. Then

$$\sigma(\mathfrak{M}) = \sum_{i=1}^{l} \boldsymbol{C}[\rho]^G (M_i - 1) = \Big(\sum_{i=1}^{n'} \boldsymbol{C}[U](X_i - 1)\Big) \cap \boldsymbol{C}[\rho]^G,$$

which is a maximal ideal. Hence we can choose such a maximal ideal $\mathfrak{N}$ of $\boldsymbol{C}[U]$ as $\mathfrak{N} \cap \boldsymbol{C}[\rho]^G = \mathfrak{M}$ and express as $\mathfrak{N} = \sum_{i=1}^{n'} \boldsymbol{C}[U](X_i - b_i)$ for some $b_i \in \boldsymbol{C}$. Let $\hat{\sigma}$ be an element of $GL(U)$ defined by $\hat{\sigma}(Y_i) = b_i^{-1} Y_i$ $(1 \leq i \leq n')$. We express $M_i - \sigma(M_i)/M_i = \sum_{j=1}^{n'} f_{ij}(X_j - b_j)$ for $1 \leq i \leq l$ and some $f_{ij} \in \boldsymbol{C}[U]$. Then $\hat{\sigma}(M_i) - \sigma(M_i)/M_i = \sum_{j=1}^{n'} \hat{\sigma}(f_{ij}) b_j (X_j - 1)$ and so $\hat{\sigma}(M_i)/M_i = \sigma(M_i)/M_i$. Consequently $\hat{\sigma}$ is the transformation desired in the assertion. Q.E.D.

Proposition 3.2. *Suppose that L is a finite subgroup of $\mathrm{Aut}_{\boldsymbol{C}\text{-}alg}(\boldsymbol{C}[\rho]^G)$ which preserves $\boldsymbol{Z}^n$-gradation. Then there is a diagonal subgroup $\tilde{L}$ of $GL_n(\boldsymbol{C})$ such that $\tilde{L} \supseteq \rho(G)$, $\mathrm{Im}(\tilde{L} \to \mathrm{Aut}_{\boldsymbol{C}\text{-}alg}(\boldsymbol{C}[\rho]^G)) = L$ and $[\tilde{L} : \rho(G)] = |L|$.*

Proof. Let σ be an element of L and u denote the order of σ. Let $\hat{\sigma}$ be a diagonal element in $GL(U)$ inducing σ and let K be the Zariski closure of the subgroup of $GL(U)$ generated by the set $\rho(G)|_U \cup \{\hat{\sigma}^u\}$ of restrictions. Clearly K is reductive. Moreover $\boldsymbol{C}[U]^K = \boldsymbol{C}[U]^{\langle \rho(G)|_U, \hat{\sigma}^u \rangle} = \boldsymbol{C}[\rho]^G$. Let x

be a point of U associated to $\sum_{i=1}^{n'} C[U](X_i - 1)$. Then $\pi_{U,G}^{-1}(\pi_{U,G}(x)) = Gx \subseteq Kx \subseteq \pi_{U,G}^{-1}(\pi_{U,G}(x))$ where $\pi_{U,G}$ denotes the quotient map $U \to U/G$, because Gx is the generic closed G-orbit. From these inclusions, it follows that $\hat{\sigma}^u = \tau|_U$ for some $\tau \in G$. Then we choose an element $\hat{\tau}$ from G in such a way that $\hat{\tau}^u = \tau$. We define a diagonal element $\tilde{\sigma}$ in $GL_n(C)$ which extends $\hat{\sigma}$ and $\tilde{\sigma}(Y_j) = \hat{\tau}(Y_j)\,(n' < j \leq n)$. Then $\tilde{\sigma}^u \in \rho(G)$. The assertion follows easily from this observation, because L is abelian. Q.E.D.

4. Proof of Theorem 1.1. Let $V(\rho)_\chi$ denote the subspace of $V(\rho)$ consisting of all vectors of a weight χ of G. Let $\{\chi_1, \ldots, \chi_m\}$ be a set consisting of all distinct weights of G which appear in $V(\rho)$.

Affine simplicial toric singularities are quotient singularities of origins of affine spaces by finite diagonal groups (cf. [3]). Thus the "if" part of Theorem 1.1 follows immediately from Proposition 3.2. So we suppose that the representation ρ of G admits a finite central coregular extension (φ, H). Let $\phi: G \to H$ be a morphism such that $\rho = \varphi \circ \phi$. Then, since $Z_H(G) = H$, φ induces a subrepresentation $\varphi_\chi: H \to GL(V(\varphi)_\chi)$. We may suppose that $n = n'$, $V(\varphi)^H = \{0\}$, $G = \phi(G)$ and φ is injective.

Lemma 4.1. *There is a finite subgroup N of H generated by pseudo-reflections in $GL(V(\varphi))$ such that $V(\varphi)/H \cong (V(\varphi)/G)/N$.*

Proof. First, suppose that $\dim V(\varphi)/G = 1$, i.e., $C[\varphi]^G = C[M]$ for some monomial M. Then G is identified with the set consisting of all diagonal matrices σ in $GL_n(C)$ satisfying $\sigma(M) = M$. By this observation and Proposition 3.2, we can choose an element τ from H which is a pseudo-reflection in $GL(V(\varphi))$ in such a way that $C[\varphi]^H = C[M]^{\langle\tau\rangle}$. Then $N = \langle \tau \rangle$ is the group desired in the assertion of this lemma.

Suppose that $\dim V(\varphi)/G \geq 2$. By purity of branch loci, we see that $\mathrm{Im}\,(H \to \mathrm{Aut}(V(\varphi)/G))$ is generated by $\{\sigma \in H \mid (\sigma - 1)(C[\varphi]^G) \subseteq \mathfrak{P}$ for a prime ideal $\mathfrak{P}$ of $C[\varphi]^G$ of height one$\}$. Then $\mathrm{Im}(H \to \mathrm{Aut}(V(\varphi)/G))$ is generated by the image of $\cup_{x \in V(\varphi)/G \setminus \pi(0)} H_x$ and consequently it is generated by the image of $\cup_\xi H_\xi$ where π denotes the quotient map $V(\varphi) \to V(\varphi)/G$ and ξ runs through the set of all closed G-orbits in $V(\varphi) \setminus \pi^{-1}(\pi(0))$. For such an element ξ, by slice étale theorem [4], the slice representation (φ_ξ, H_ξ) of (φ, H) at ξ is coregular, and moreover (φ, H_ξ) is coregular, because the adjoint representation of H is trivial. Since $Z_{H_\xi}(G_\xi) = H_\xi$, we have $Z_{H_\xi}(H_\xi^o) = H_\xi^o$. Thus we inductively see that there is a finite subgroup N^ξ of H_ξ generated by pseudo-reflections in $GL(V(\varphi))$ such that $(V(\varphi)/H_\xi^o)/N^\xi \cong V(\varphi)/H_\xi$. Then $(V(\varphi)/G)/N^\xi \cong ((V(\varphi)/H_\xi^o)/G)/N^\xi \cong ((V(\varphi)/H_\xi^o)/N^\xi)/G \cong (V(\varphi)/H_\xi)/G \cong (V(\varphi)/G)/H_\xi$, which implies $\mathrm{Im}(N^\xi \to \mathrm{Aut}(V(\varphi)/G)) = \mathrm{Im}(H_\xi \to \mathrm{Aut}(V(\varphi)/G))$. Since H/G is finite, for some k we have finite subgroups $L_i\ (1 \leq i \leq k)$ of H generated by pseudo-reflections in $GL(V(\varphi))$ and $V(\varphi)/H \cong V(\varphi)/G \cdot \langle \cup_{i=1}^k L_i \rangle$. Since the factor group $(\varphi_{\chi_i}(G \cdot \langle \cup_{i=1}^k L_i \rangle) \cap SL(V(\varphi)_{\chi_i}))/(\varphi_{\chi_i}(G) \cap SL(V(\varphi)_{\chi_i}))$ is finite, $\varphi_{\chi_i}(G \cdot \langle \cup_{i=1}^k L_i \rangle) \cap SL(V(\varphi)_{\chi_i})$ is also finite. Thus $N = \langle \cup_{i=1}^k L_i \rangle$ is the group desired in the assertion of this lemma. Q.E.D.

Let N be a subgroup stated in Lemma 4.1. Then $N = N_1 \times \cdots \times N_m$

for some finite subgroups N_i generated by pseudo-reflections in $GL(V(\rho)_{\chi_i})$. Clearly $\varphi_{\chi_j}(N_i) = 1 (i \neq j)$. We may assume that $H = G \cdot N$ and all χ_i's are nontrivial. Put $I = \{1, \ldots, m\}$ and let $\{f_{ij} \mid 1 \leq j \leq n_i\}$ be a regular system of homogeneous parameters of $C[\varphi_{\chi_i}]^{N_i}$. For a monomial $M = \Pi_{i,j} f_{ij}^{a_{ij}}$ of $\{f_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n_i\}$ in $C[\varphi]$, let $\mathrm{Supp}M = \{(i, j) \mid a_{ij} > 0\}$ and, moreover, put $\mathrm{supp}M = \{i \mid a_{ij} > 0 \text{ for some } j\}$.

First we suppose that, for some k, the canonical map

$$(V(\oplus_{i \in I \setminus \{k\}} \varphi_{\chi_i}) / N \cdot \mathrm{Ker}(\chi_k))^G \to V(\oplus_{i \in I \setminus \{k\}} \varphi_{\chi_i}) / H$$

is surjective and prove the assertion by induction on $\dim G$. Put $B = C[\oplus_{i \in I \setminus \{k\}} \varphi_{\chi_i}]$ and let $\{g_1, \ldots, g_t\}$ be a minimal (may be empty) system of homogeneous generators of B^H. Obviously this system can be extended to a minimal system of homogeneous generators of $B^{N \cdot \mathrm{Ker}\chi_k}$. We naturally define the non-negative gradations on $C[\varphi_{\chi_k}]$ and $B^{N \cdot \mathrm{Ker}\chi_k} / (\sum_{j=1}^t B^{N \cdot \mathrm{Ker}\chi_k} \cdot g_j)$ which are induced by the action of $G / \mathrm{Ker}(\chi_k) \cong C^*$ and preserved by the action of N such that

$$C[\varphi_{\chi_k}]^{N_k} \#_C \left(B^{N \cdot \mathrm{Ker}\chi_k} / \left(\sum_{j=1}^t B^{N \cdot \mathrm{Ker}\chi_k} \cdot g_j\right)\right) \cong C[\varphi]^H / \left(\sum_{j=1}^t C[\varphi]^H \cdot g_j\right).$$

From this isomorphism we infer that $\mathrm{ht}(\sum_{j=1}^t B^{N \cdot \mathrm{Ker}\chi_k} \cdot g_j) = t$ and that $C[\varphi_{\chi_k}]^{N_k} \#_C (B^{N \cdot \mathrm{Ker}\chi_k} / (\sum_{j=1}^t B^{N \cdot \mathrm{Ker}\chi_k} \cdot g_j))$ is a polynomial ring, which implies that $C[f_{k1}] \#_C (B^{N \cdot \mathrm{Ker}\chi_k} / (\sum_{j=1}^t B^{N \cdot \mathrm{Ker}\chi_k} \cdot g_j))$ is also a polynomial ring. By Proposition 2.3, we see that $(\oplus_{i \in I \setminus \{k\}} \varphi_{\chi_i}, \mathrm{Ker}(\chi_k)^0)$ admits a finite central coregular extension and hence by induction hypothesis we can choose a system $\{x_1, \ldots, x_s\}$ of homogeneous parameters of $B^{\mathrm{Ker}\chi_k}$ consisting of monomials in B. We may assume $C[x_1, \ldots, x_s]^{G/\mathrm{Ker}\chi_k} = C[x_1, \ldots, x_{t'}]$ for some $t' \leq s$. Since $n = n'$ and each $g_i (1 \leq i \leq t)$ is integral over $C[x_1, \ldots, x_{t'}]$, the following inequalities hold;

$$s > t' = \mathrm{ht}\left(\sum_{j=1}^{t'} B^{\mathrm{Ker}(\chi_k)^0} \cdot x_j\right) \geq \mathrm{ht}\left(\sum_{j=1}^{t} B^{N \cdot \mathrm{Ker}(\chi_k)^0} \cdot g_j\right) = t.$$

On the other hand, by Lemma 2.2, we see that one of $\dim B^{N \cdot \mathrm{Ker}\chi_k} - t$ and $\dim V(\varphi_{\chi_k})$ is equal to 1. Since the proof is similar, one needs to treat only the case where $\dim V(\varphi_{\chi_k}) > 1$. Then we must have $t' = t = s - 1$. We see that $C[\varphi]^G$ is simplicial as an affine semigroup ring, because it is integral over $(C[\varphi_{\chi_k}] \otimes_C C[x_1, \ldots, x_s])^{G/\mathrm{Ker}\chi_k}$.

Next we suppose that

$$(V(\oplus_{j \in I \setminus \{i\}} \varphi_{\chi_j}) / \mathrm{Ker}(\chi_i) \cdot N)^G \to V(\oplus_{j \in I \setminus \{i\}} \varphi_{\chi_j}) / H$$

are not surjective for all $1 \leq i \leq m$. Fix an arbitrary index s with $1 \leq s \leq m$ and let $\{M_1, \ldots, M_l\}$ be a minimal system of homogeneous generators of $C[\oplus_{j \in I \setminus \{s\}} \varphi_{\chi_j}]^{N \cdot \mathrm{Ker}\chi_s}$ consisting of monomials of $\{f_{ij}\}$. We may assume that $\{M_1, \ldots, M_{l'}\}$ is a subset of $\{M_1, \ldots, M_l\}$ consisting of M_i with minimal $\mathrm{Supp}M_i$. Then $C[M_1, \ldots, M_l]$ is integral over $C[M_1, \ldots, M_{l'}]$. We may assume that $(C[\varphi_{\chi_s}]^N \otimes_C C[M_1])^{G/\mathrm{Ker}\chi_s} \neq C[M_1]^{G/\mathrm{Ker}\chi_s}$. Since $C[M_i \mid 1 \leq i \leq l, \mathrm{Supp}M_i \subseteq \mathrm{Supp}M_1] = C[f_{ij} \mid (i, j) \in \mathrm{Supp}M_1]^{\mathrm{Ker}\chi_s}$ and it is integral over $C[M_1]$, this algebra coinsides with $C[M_1]$. Hence a minimal system of homogeneous generators of $(C[\varphi_{\chi_s}]^N \otimes_C C[M_1])^{G/\mathrm{Ker}\chi_s}$ can be extended to a regular system of homogeneous parameters of $C[\varphi]^H$. Let

$\{L_1, \ldots, L_k\}$ be a unique regular system of homogeneous parameters of $C[\varphi]^H$ consisting of monomials of $\{f_{ij}\}$. We easily have $\dim(C[\varphi_{\chi_s}]^N \otimes_C C[M_1, \ldots, M_{l'}])^{G/\mathrm{Ker}\chi_s} = \dim V(\varphi_{\chi_s}) + \dim C[\oplus_{j\in I\setminus\{s\}} \varphi_{\chi_j}]^{\mathrm{Ker}\chi_s} - 1 = \dim V(\varphi_{\chi_s}) + \dim C[\oplus_{j\in I\setminus\{s\}} \varphi_{\chi_j}]^H$. By this and $\dim(C[\varphi_{\chi_s}]^N \otimes_C C[M_1])^{G/\mathrm{Ker}\chi_s} = \dim V(\varphi_{\chi_s})$, we infer that $\{L_i \mid s \in \mathrm{supp}L_i\}$ is a regular system of homogeneous parameters of $(C[\varphi_{\chi_s}]^N \otimes_C C[M_1])^{G/\mathrm{Ker}\chi_s}$. Moreover M_1 is a unique element M_i in $\{M_1, \ldots, M_{l'}\}$ such that $(C[\varphi_{\chi_s}]^N \otimes_C C[M_i])^{G/\mathrm{Ker}\chi_s} \neq C[M_i]^{G/\mathrm{Ker}\chi_s}$. On the other hand, suppose that $n_u > 1$ for some $u \in \mathrm{supp}M_1$. Let M_{11} be a monomial of $\{f_{uj} \mid 1 \leq j \leq n_u\}$ and M_{12} a monomial of $\{f_{ij} \mid i \neq u\}$ which satisfy $M_1 = M_{11}M_{12}$. Since $M_{11} \in C[\varphi_{\chi_u}]_{\chi_u^{-\deg M_{11}}}$ and $n_u > 1$, we can choose a monomial $\tilde{M}_1$ of $\{f_{uj} \mid 1 \leq j \leq n_u\}$ such that an irreducible divisor of M_{11} does not divide $\tilde{M}_1$ in $C[\varphi_{\chi_u}]^N$ and $\deg M_{11}$ is a divisor of $\deg\tilde{M}_1$. Then $\tilde{M}_1 M_{12}^{\deg\tilde{M}_1/\deg M_{11}} \in C[\oplus_{j\in I\setminus\{s\}} \varphi_{\chi_j}]^{\mathrm{Ker}\chi_s}$ and $(C[\varphi_{\chi_s}]^N \otimes_C C[\tilde{M}_1 M_{12}^{\deg\tilde{M}_1/\deg M_{11}}])^G \neq C[\tilde{M}_1 M_{12}^{\deg\tilde{M}_1/\deg M_{11}}]^G$. Thus a multiple of $\tilde{M}_1$ is divisible by M_{11} in $C[\varphi_{\chi_u}]^N$, which is a contradiction. We must have $\dim V(\varphi_{\chi_j}) = 1$ for any $j \in I_s$, where $I_s = \mathrm{supp}M_1$. Moving s, we consequently define (nonempty) subsets I_s as above for each $1 \leq s \leq m$. Since $\{s\} \cup I_s = \cup_{s\in\mathrm{supp}L_i} \mathrm{supp}L_i$, we see the fact that $I_s \ni s'$ implies the fact that $I_{s'} \ni s$. Thus $\cup_{1\leq s\leq m} I_s = \{1, \ldots, m\}$, and the associated cone of $V(\rho)/G$ is simplicial. The proof of Theorem 1.1 has just been completed.

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