64. *On Foliation on Complex Spaces*

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§0. Introduction. In this paper, we discuss foliations on reduced complex spaces. On complex manifolds, foliations are defined in two ways: as coherent subsheaves of the sheaf Θ of germs of holomorphic vector fields and of the sheaf Ω of germs of holomorphic 1-forms, satisfying the "integrability conditions". Foliations defined by vector fields and by 1-forms correspond with each other (cf. [1],[5],[6]). We define foliations on complex spaces in two ways, using vector fields and 1-forms, as a natural extension of the cases on manifolds (Definition 1.0). As the case on a complex manifolds, these two definitions are essentially equivalent with each other (Theorem 1.5). We investigate effects of morphisms of complex spaces on foliations on them. Let $X \to Y$ be a proper modification of reduced complex spaces. Then foliations on X and on Y are correspondent with each other (Theorem 3.3). Thus foliations are bimeromorphically invariant. Details of proofs etc. are written in [4].

§1. Coherent foliations on complex spaces. Let $(X, \mathcal{O}_X)$ be a reduced complex space. We use the following notations:

Ω_X: the sheaf of germs of holomorphic 1-forms on X

Θ_X: the sheaf of germs of holomorphic vector fields on X

sp$X$: the underlying topological space of the complex space X.

By definition, $\Theta_X = \Omega_X^*$: the dual of Ω_X. If X is a closed complex subspace of a domain $D \subset C^m$ defined by a coherent $\mathcal{O}_D$-ideal $\mathcal{I}$, note that $\Omega_X = (\Omega_D / \mathcal{O}_D d\mathcal{I}) \mid_X$.

For a coherent $\mathcal{O}_X$-module $\mathcal{S}$, we set

$$\mathrm{Sing}\mathcal{S} := \{x \in X \mid \mathcal{S}_x \text{ is not } \mathcal{O}_{X,x}\text{-free}\}.$$

If the complex space X is reduced, then Sing $\mathcal{S}$ is a thin analytic set in X. For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, we use the notation:

$$S(\mathcal{T}) := \mathrm{Sing}\,\mathcal{S} \cup \mathrm{Sing}(\mathcal{S}/\mathcal{T}).$$

$S(\mathcal{T})$ is an analytic set in X satisfying

$$S(\mathcal{T}) \supset \mathrm{Sing}\,\mathcal{T}.$$

On $X - S(\mathcal{T})$, $\mathcal{T}$ is locally a direct summand of $\mathcal{S}$.

Note that

$$\mathrm{Sing}\,X = \mathrm{Sing}\,\Omega_X$$

holds, where Sing X is the singular locus of the complex space X.

Definition 1.0. We define coherent foliations in two ways.

• Definition a) (by 1-forms).

0) A *coherent foliation* on X is a coherent $\mathcal{O}_X$-submodule F of Ω_X satisfying

(1.1) $$dF_x \subset F_x \wedge \Omega_{X,x}$$

at any $x \in X - S(F)$. This condition is called the *integrability condition.* We call $S(F)$ the *singular locus* of the foliation F.

1) A coherent foliation $F \subset \Omega_X$ is said to be *reduced* if, for any open subspace $U \subset X$, $\xi \in \Gamma(U, \Omega_X)$ and $\xi|_{U-S(F)} \in \Gamma(U - S(F), F)$ imply $\xi \in \Gamma(U, F)$.

• Definition b) (by vector fields).

0) A *coherent foliation* on X is a coherent $\mathcal{O}_X$-submodule E of Θ_X satisfying

$$[E_x, E_x] \subset E_x \tag{1.2}$$

at any $x \in X - (S(E) \cup \operatorname{Sing} X)$. This condition is called the *integrability condition.* We call $S(E) \cup \operatorname{Sing} X$ the *singular locus* of the foliation E.

1) A coherent foliation $E \subset \Theta_X$ is said to be *reduced* if, for any open subspace $U \subset X$, $v \in \Gamma(U, \Theta_X)$ and $v|_{U-(S(E) \cup \operatorname{Sing} X)} \in \Gamma(U - (S(E) \cup \operatorname{Sing} X), E)$ imply $v \in \Gamma(U, E)$.

Remarks. 0) If the complex space $(X, \mathcal{O}_X)$ is a complex manifold, then $\operatorname{Sing} X = \operatorname{Sing} \Omega_X$ is empty and the above definitions are those of holomorphic foliations with singularities in [5],[6] and [1].

1) The reducedness of foliations $F \subset \Omega_X$ and $E \subset \Theta_X$ is just the fullness (Definition 2.4 in §2 below) of the coherent submodules F and E in Ω_X and Θ_X, respectively.

2) The above integrability condition (1.1) for F is equivalent to the following:

$$\xi \in F_x,\ u \in F_x^a,\ v \in F_x^a \Rightarrow d\xi(u, v) = 0 \in \mathcal{O}_{X,x} \tag{1.3}$$

holds at any $x \in X$.

3) If $E \subset \Theta_X$ is a reduced coherent foliation, then the condition (1.2) holds at *any* $x \in X$. This is a direct consequence of the fullness of E in Θ_X.

There are natural correspondences between foliations defined by the above two definitions. Namely,

Definition 1.4. 0) For a coherent foliation $F \subset \Omega_X$, we define a coherent foliation $F^a \subset \Theta_X$ by

$$F_x^a := \{v \in \Theta_{X,x} \mid \langle v, \xi \rangle = 0 \text{ for all } \xi \in F_x\}.$$

1) For a coherent foliation $E \subset \Theta_X$, we define a coherent foliation $E^\perp \subset \Omega_X$ by

$$E_x^\perp := \{\xi \in \Omega_{X,x} \mid \langle v, \xi \rangle = 0 \text{ for all } v \in E_x\}.$$

Theorem 1.5. 0) *For a coherent foliation $F \subset \Omega_X$, the $\mathcal{O}_X$-submodules F^a of Θ_X and $F^{a\perp} = (F^a)^\perp$ of Ω_X are reduced coherent foliations.*

1) *For a coherent foliation $E \subset \Theta_X$, the $\mathcal{O}_X$-submodules $E^\perp$ of Ω_X and $E^{\perp a} = (E^\perp)^a$ of Θ_X are reduced coherent foliations.*

2) *These correspondences*

$$F \subset \Omega_X \to F^a \subset \Theta_X \text{ and } E \subset \Theta_X \to E^\perp \subset \Omega_X$$

restricted to reduced coherent foliations are inverse of each other.

§2. Outline of the proof of Theorem 1.5. Let $(X, \mathcal{O}_X)$ be a reduced complex space and $\mathcal{S}$ a coherent $\mathcal{O}_X$-module. To prove Theorem 1.5, we generalize Definition 1.4.

Definition 2.0. Let $\mathcal{S}$ be a coherent $\mathcal{O}_X$-module and $\mathcal{S}^*$ the dual $\mathcal{O}_X$-module of $\mathcal{S}$.

0) For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, an $\mathcal{O}_X$-submodule $\mathcal{T}_F$ of $\mathcal{S}$ is defined by a complete presheaf

$$\mathcal{T}_F(U) := \{\xi \in \mathcal{S}(U) \mid \xi\,|_{U-S(\mathcal{S})} \in \mathcal{T}(U - S(\mathcal{S}))\} \text{ for } U \subset X.$$

The coherence of $\mathcal{T}_F$ is assured in Proposition 2.2 below.

1) For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, $\mathcal{T}^a$ is a coherent $\mathcal{O}_X$-submodule of $\mathcal{S}^*$ whose stalk on $x \in X$ is

$$\mathcal{T}^a_x := \{v \in \mathcal{S}^*_x \mid \langle v, \xi\rangle = 0 \text{ for all } \xi \in \mathcal{T}_x\}.$$

2) For a coherent $\mathcal{O}_X$-submodule $\mathcal{R}$ of $\mathcal{S}^*$, we define a coherent $\mathcal{O}_X$-submodule $\mathcal{R}^\perp$ of $\mathcal{S}$ by

$$\mathcal{R}^\perp_x := \{\xi \in \mathcal{S}_x \mid \langle v, \xi\rangle = 0 \text{ for all } v \in \mathcal{R}_x\}.$$

Remarks. 0) Since $\mathcal{T}$ is coherent, the quotient $\mathcal{O}_X$-module $\mathcal{Q} = \mathcal{S}/\mathcal{T}$ is also coherent. It follows that, as $\mathcal{O}_X$-modules, $\mathcal{T}^a \simeq \mathcal{Q}^*$, where $\mathcal{Q}^*$ is the dual of the quotient module $\mathcal{Q}$. Thus $\mathcal{T}^a$ is coherent and

$$S(\mathcal{T}^a) \subset S(\mathcal{T}).$$

1) $\mathcal{R}^\perp$ is not always isomorphic to the $\mathcal{O}_X$-submodule $\mathcal{R}^a$ of the bidual $\mathcal{S}^{**}$ of $\mathcal{S}$. It is the inverse image of $\mathcal{R}^a$ by the canonical $\mathcal{O}_X$-morphism $\mathcal{S} \to \mathcal{S}^{**}$. Thus $\mathcal{R}^\perp$ is the kernel of the composite $\mathcal{O}_X$-morphism

$$\mathcal{S} \to \mathcal{S}^{**} \to \mathcal{S}^{**}/\mathcal{R}^a$$

and the coherence of $\mathcal{R}^\perp$ follows from that of $\mathcal{S}^{**}/\mathcal{R}^a$. Note that $\mathcal{S}^{**}/\mathcal{R}^a$ is coherent if and only if $\mathcal{R}^a$ is, and that the coherence of $\mathcal{R}^a$ is induced from that of $\mathcal{R}$.

Definition 2.1. 0) For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, we define a coherent $\mathcal{O}_X$-submodule $\mathcal{T}^{a\perp}$ of $\mathcal{S}$ by

$$\mathcal{T}^{a\perp} := (\mathcal{T}^a)^\perp.$$

1) For a coherent $\mathcal{O}_X$-submodule $\mathcal{R}$ of $\mathcal{S}^*$, we define a coherent $\mathcal{O}_X$-submodule $\mathcal{R}^{\perp a}$ of $\mathcal{S}^*$ by

$$\mathcal{R}^{\perp a} := (\mathcal{R}^\perp)^a.$$

These submodules have the following properties.

Proposition 2.2. *Let $\mathcal{S}$ be a coherent $\mathcal{O}_X$-module and $\mathcal{S}^*$ the dual $\mathcal{O}_X$-module of $\mathcal{S}$.*

0) *For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$, $\mathcal{T}_F$ satisfies*

a) $\mathcal{T} \subset \mathcal{T}_F$.

b) $\mathcal{T}\,|_{X-S(\mathcal{T})} = \mathcal{T}_F\,|_{X-S(\mathcal{T})}$.

c) *If a section $\xi \in \Gamma(U, \mathcal{S})$ on an open subset $U \subset X$ satisfies, for some thin analytic set A in U, $\xi\,|_{U-A} \in \Gamma(U-A, \mathcal{T})$, then $\xi \in \Gamma(U, \mathcal{T}_F)$.*

1) *For a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$,*

$$\mathcal{T}_F = \mathcal{T}^{a\perp}.$$

Thus $\mathcal{T}_F$ is coherent.

2) *For a coherent $\mathcal{O}_X$-submodule $\mathcal{R}$ of $\mathcal{S}^*$,*

$$\mathcal{R}_F = \mathcal{R}^{\perp a}.$$

Corollary 2.3. *If two coherent $\mathcal{O}_X$-submodules $\mathcal{T}_0$ and $\mathcal{T}_1$ of $\mathcal{S}$ satisfy*

$$\mathcal{T}_0\,|_{X-A} = \mathcal{T}_1\,|_{X-A}$$

for a thin analytic set A in X, then

$$(\mathcal{T}_0)_F = (\mathcal{T}_1)_F.$$

The following are easily verified.

$$(\mathcal{T}_F)_F = (\mathcal{T}^{a\perp})^{a\perp} = \mathcal{T}^{a\perp} = \mathcal{T}_F$$
$$(\mathcal{R}_F)_F = (\mathcal{R}^{\perp a})^{\perp a} = \mathcal{R}^{\perp a} = \mathcal{R}_F.$$

Definition 2.4. Let $\mathcal{S}$ be a coherent $\mathcal{O}_X$-module. A coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$ is said to be *full* in $\mathcal{S}$ if it satisfies

$$\mathcal{T} = \mathcal{T}_F.$$

Corollary 2.5. *Let $\mathcal{S}$ be a coherent $\mathcal{O}_X$-module and $\mathcal{S}^*$ the dual $\mathcal{O}_X$-module of $\mathcal{S}$.*

0) *A coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$ is full in $\mathcal{S}$ if and only if $\mathcal{T}^{a\perp} = \mathcal{T}$.*

1) *A coherent $\mathcal{O}_X$-submodule $\mathcal{R}$ of $\mathcal{S}^*$ is full in $\mathcal{S}^*$ if and only if $\mathcal{R}^{\perp a} = \mathcal{R}$.*

Let us summarize the attributes of the correspondences $\cdot^a$ and $\cdot^\perp$.

Theorem 2.6. a) *By the correspondence $\mathcal{T} \subset \mathcal{S} \to \mathcal{T}^a \subset \mathcal{S}^*$, a coherent $\mathcal{O}_X$-submodule $\mathcal{T}$ of $\mathcal{S}$ is corresponded to the coherent $\mathcal{O}_X$-submodule $\mathcal{T}^a$ of $\mathcal{S}^*$, which is full in $\mathcal{S}^*$. This correspondence inverts the including relations.*

b) *By the correspondece $\mathcal{R} \subset \mathcal{S}^* \to \mathcal{R}^\perp \subset \mathcal{S}$, a coherent $\mathcal{O}_X$-submodule $\mathcal{R}$ of $\mathcal{S}^*$ is corresponded to the coherent $\mathcal{O}_X$-submodule $\mathcal{R}^\perp$ of $\mathcal{S}$, which is full in $\mathcal{S}$. This correspondece inverts the including relations.*

c) *These correspondences, restricted to full submodules, are the inverse of each other.*

Proof of Theorem 1.5. Except for the integrability conditions, the assertions are direct consequences of Theorem 2.6. The integrability condition (1.3) is easily verified.

§3. Proper modifications and foliations. Let $f : X \to Y$ be a morphism of reduced complex spaces. For a coherent $\mathcal{O}_Y$-submodule F of Ω_Y, a coherent $\mathcal{O}_X$-submodule $f^\star F$ of Ω_X, is determined by means of pulling back of 1-forms. Note that $f^{-1}(S(F))$ is an analytic set in X and that

$$\operatorname{Sing} X = \operatorname{Sing} \Omega_X \subset S(f^\star F) \subset f^{-1}(S(F)).$$

In the case that $f^{-1}(S(F))$ is thin in X, it follows from the equivalence of conditions (1.1) and (1.3) that

Proposition and Definition 3.0. *Let $f : X \to Y$ be a morphism of reduced complex spaces and $F \subset \Omega_Y$ a coherent foliation on Y. Assume $f^{-1}(S(F))$ is thin in X. Then the $\mathcal{O}_X$-submodule $f^\star F \subset \Omega_X$ is a coherent foliation on X. We call it the* ***pulled-back*** *of F by f.*

Now we investigate the case that X and Y are reduced and that $f : X \to Y$ is a proper modification. First, we recall the following theorem (cf. eg. [2]).

Theorem 3.1 (Lifting lemma) ([2] pp.154-155). *Assume that $f : X \to Y$ is a proper modification of a reduced complex space Y. Then f induces an isomorphism $\tilde{f} : \mathcal{M}_Y \to f_*\mathcal{M}_X$ which makes the following diagram commute:*

$$\begin{array}{ccc} \mathcal{O}_Y & \xrightarrow{\tilde{f}} & f_*\mathcal{O}_X \\ \cap & & \cap \\ \mathcal{M}_Y & \underset{\tilde{f}}{\overset{\cong}{\to}} & f_*\mathcal{M}_X \end{array}.$$

Let $f : X \to Y$ be a proper modification of a reduced complex space Y. The above theorems tell us that the direct image $f_*\mathcal{O}_X$ is an $\mathcal{O}_Y$-coherent

$\mathcal{O}_Y$-submodule of $f_*\mathcal{M}_X \simeq \mathcal{M}_Y$. On $U \subset Y$, a section $v \in \Gamma(U, f_*\Theta_X) = \Gamma(f^{-1}(U), \Theta_X)$ defines a derivation $\mathcal{O}_Y|_U \to f_*\mathcal{O}_X|_U$, i.e. the composite with $\tilde{f} : \mathcal{O}_Y|_U \to f_*\mathcal{O}_X|_U$. Thus v defines an $\mathcal{O}_Y|_U$-morphism $\Omega_Y|_U \to f_*\mathcal{O}_X|_U$. This, followed by $f_*\mathcal{O}_X \subset f_*\mathcal{M}_X \simeq \mathcal{M}_Y$, determines a unique section f_*v on U of $\mathcal{H}om_{\mathcal{O}_Y}(\Omega_Y, \mathcal{M}_Y) \simeq \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{M}_Y$. Since $\mathcal{H}om_{\mathcal{O}_Y}(\Omega_Y, \cdot)$ is left exact, this $\mathcal{O}_Y$-morphism $\mathcal{H}om_{\mathcal{O}_Y}(\Omega_Y, f_*\mathcal{O}_X) \to \mathcal{H}om_{\mathcal{O}_Y}(\Omega_Y, \mathcal{M}_Y) \simeq \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{M}_Y$ is injective. Thus an $\mathcal{O}_X$-coherent $\mathcal{O}_X$-submodule E of Θ_X determines a unique $\mathcal{O}_Y$-coherent $\mathcal{O}_Y$-submodule of $\Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{M}_Y$. The inverse image of it under $\Theta_Y \to \Theta_Y \otimes_{\mathcal{O}_Y} \mathcal{M}_Y$ is an $\mathcal{O}_Y$-coherent $\mathcal{O}_Y$-submodule of Θ_Y, which we denote by $f_\star E$. If $E \subset \Theta_X$ is a coherent foliation on X, then $f_\star E \subset \Theta_Y$ is a coherent foliation on Y.

Definition 3.2. For a coherent foliation $E \subset \Theta_X$ on X, we call the coherent foliation

$$f_\star E \subset \Theta_Y$$

on Y the *pushed-forward* of E by f.

Since f is biholomorphic outside of thin analytic sets, we have

Theorem 3.3. *Let $f : X \to Y$ be a proper modification of a reduced complex space Y. Then the correspondences*

$$F \subset \Omega_Y \to (f^\star F)^a \subset \Theta_X \text{ and } E \subset \Theta_X \to (f_\star E)^\perp \subset \Omega_Y$$

are, restricted to reduced coherent foliations, the inverse of each other.

Thus, under proper modifications, reduced coherent foliations are "essentially invariant". We can investigate reduced coherent foliations on a reduced complex space X by investigating the pulled-back of them by a desingularization $\tilde{X} \to X$ of X.

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