61. *Mordell-Weil Lattices for Higher Genus Fibration*

By Tetsuji SHIODA

Department of Mathematics, Rikkyo University

(Communicated by Kunihiko KODAIRA, M. J. A., Nov. 12, 1992)

1. Introduction. The notion of the Mordell-Weil lattice of an elliptic curve over a function field (or of an elliptic surface) has been established in our previous work together with its basic properties (see [4], [5]). In this note, we sketch a generalization to the case of an algebraic curve of higher genus over a function field (or of an algebraic surface with higher genus fibration), and give a nontrivial example. Detailed account is in preparation.

Let $K = k(C)$ be the function field of an algebraic curve C over an algebraically closed ground field k; the curve C should serve as the base curve of some fibration and it is assumed to be smooth and projective. Let Γ/K be a smooth projective curve of genus $g > 0$ with a K-rational point $O \in \Gamma(K)$, and let J/K denote the Jacobian variety of Γ/K. Assume the following condition:

$(*)$ The K/k-trace of J is trivial.

Then the group of K-rational points $J(K)$ is a finitely generated abelian group (Mordell-Weil theorem), and the set $\Gamma(K)$ of K-rational points of Γ is a finite subset of $J(K)$ if $g > 1$ (Mordell conjecture for function fields = Theorem of Grauert-Manin-Samuel). We refer to Lang's book [2] for the above.

The main idea of this note is *to view the Mordell-Weil group $J(K)$ (modulo torsion) as a Euclidean lattice* with respect to a natural pairing defined in terms of intersection theory on an associated surface, in the same way as the case of $g = 1$ (cf. [4], [5]).

2. Basic theorems. Given Γ/K as above, we can associate an algebraic surface with a relatively minimal fibration:

$$f : S \to C. \tag{1}$$

Namely, S is a smooth projective surface, f is a morphism with the generic fibre Γ/K and there are no exceptional curves of the first kind in any fibre. The K-rational points of Γ are in a natural one-one correspondence with the sections of f; for $P \in \Gamma(K)$, (P) will denote the section regarded as a curve in S. Let $\mathrm{NS}(S)$ be the Néron-Severi group of S. Then we have

Theorem 2.1. *Under the assumption* $(*)$, *there is a natural isomorphism*:

$$J(K) \simeq \mathrm{NS}(S)/T \tag{2}$$

where T is the subgroup generated by (O) and all the irreducible components of fibres of f.

For simplicity, assume in the following that $(**)$ $\mathrm{NS}(S)$ is torsion-free. Then it forms an integral lattice with respect to the intersection pairing, of signature $(1, \rho - 1)$ (Hodge index theorem), $\rho = \mathrm{rk}\ \mathrm{NS}(S)$ being the Picard number of S.

Proposition 2.2. *Under the assumptions* ($*$), ($**$), T *forms a sublattice of* $\mathrm{NS}(S)$, *which is decomposed as follows*:

(3) $$T = U \oplus \oplus_{v \in R} T_v,$$

where U *is a rank* 2 *unimodular lattice spanned by* (O), F (F *any fibre*) *and* $R = \{v \in C(k) \mid f^{-1}(v)$ *is reducible*$\}$: *for each* $v \in R$, T_v *is a negative-definite sublattice spanned by the irreducible components of* $f^{-1}(v)$ *which do not intersect the zero section* (O).

Following the terminology for the case $g = 1$, we call T the *trivial sublattice* of $\mathrm{NS}(S)$ and $L = T^{\perp}$ the *essential sublattice*. Observe that L is a negative-definite lattice of rank r. We note that T_v is *not* a root lattice in general if $g > 1$ in contrast to the elliptic case $g = 1$. In any case, (2) and (3) imply the formula (cf. [8]):

(4) $$\rho = r + 2 + \sum_{v \in R} (m_v - 1), \; r = \mathrm{rk}\, J(K)$$

where m_v denotes the number of irreducible components of $f^{-1}(v)$.

Lemma 2.3. *There is a unique map which "splits" the isomorphism* (2), *i.e.*

(5) $$\varphi : J(K) \to \mathrm{NS}(S) \otimes \boldsymbol{Q}$$

such that for any $P \in J(K)$ *we have*

(6) $$\varphi(P) \equiv D_P \bmod T \otimes \boldsymbol{Q}, \; \varphi(P) \perp T$$

where D_P *is a horizontal divisor on* S *corresponding to* $P \in J(K) = \mathrm{Pic}^0(\Gamma)(K)$ *under* (2); *for instance, we can take* $D_P = (P)$ *for* $P \in \Gamma(K) \subset J(K)$. *This map is a group homomorphism such that*

(7) $$\mathrm{Ker}(\varphi) = J(K)_{tor}, \quad \mathrm{Im}(\varphi) \subset L^*$$

Theorem 2.4. *Define a symmetric bilinear form on the Mordell-Weil group* $J(K)$ *by*

(8) $$\langle P, Q \rangle = -(\varphi(P) \cdot \varphi(Q)) \in \boldsymbol{Q} \qquad (P, Q \in J(K)).$$

Then it defines the structure of a positive-definite lattice on $J(K)/J(K)_{tor}$, *which will be called the Mordell-Weil lattice of* J/K.

The map φ has an explicit description (as in [5]) which gives an explicit formula for the height pairing: if we normalize D_P so that $(D_P \cdot F) = 1$, then

(9) $$\langle P, Q \rangle = -(O^2) - (D_P \cdot D_Q) + (D_P O) + (D_Q O) - \sum_{v \in R} contr_v(P, Q)$$

(10) $$\langle P, P \rangle = -(O^2) - (D_P^2) + 2(D_P O) - \sum_{v \in R} contr_v(P)$$

with the same notation as in [4], [5]; the local contribution term $contr_v(P, Q)$ is a certain non-negative rational number and $contr_v(P) = contr_v(P, P)$.

To define the narrow Mordell-Weil lattice of J/K, let $J(K)^0$ be the subgroup of $J(K)$ which is the image of the essential sublattice $L = T^{\perp} \subset \mathrm{NS}(S)$ under (2). For any $P \in J(K)^0$, we have $contr_v(P, Q) = contr_v(P) = 0$.

Theorem 2.5. *With respect to the height pairing,* $J(K)^0$ *is a positive-definite integral lattice, isomorphic to the opposite lattice* L^- *of* L. *It will be called the narrow Mordell-Weil lattice of* J/K.

As in the elliptic case, the Mordell-Weil lattice embeds into the dual lattice of the narrow one, and we have the following (cf. [5]):

Theorem 2.6. *Suppose that the Néron-Severi lattice of S is unimodular, i.e.* $\det \mathrm{NS}(S) = \pm 1$. *Then the Mordell-Weil lattice of J/K is isomorphic to the dual lattice of the narrow Mordell-Weil lattice $J(K)^0$.*

3. Example. As the reader must be aware, the above theory of Mordell-Weil lattices (MWL) for higher genus fibration is quite parallel to that for the elliptic case treated before, and the proof is not so much different from that case. Somewhat different is the maturity of the theory of elliptic surfaces due to Kodaira [1], including well-studied facts on singular fibres (see also Néron [3], Tate [9]), which is yet to be seen for higher genus case.

So let us give a non-trivial example which settles one of the predictions made in our previous paper [6]: Introduction, p.676, paragraph (1). For any positive integer g, consider the hyperelliptic curve $\Gamma = \Gamma_\lambda$ of genus g over $K = k(t)$, defined by the following equation:

(11)
$$y^2 = x^{2g+1} + p_2 x^{2g-1} + \cdots + p_{2g}x + p_{2g+1} + t^2,\ \lambda = (p_2, \ldots, p_{2g+1}) \in k^{2g}.$$

Let $O \in \Gamma(K)$ be the (unique) point at infinity. For simplicity, assume that k has characteristic 0. (N.B. This equation is known to define a semi-universal deformation of A_n-singularity for $n = 2g : y^2 = x^{2g+1} + t^2$, with parameter λ.)

Let $u_1, \ldots, u_{2g}, u_{2g+1}$ be the roots of the algebraic equation:

(12)
$$x^{2g+1} + p_2 x^{2g-1} + \cdots + p_{2g}x + p_{2g+1} = 0.$$

There are $2(2g+1)$ K-rational points of Γ:

(13) $P_i : x = u_i,\ y = t$ and $P_i' : x = u_i,\ y = -t$ $(i = 1, \ldots, 2g+1)$,

which satisfy the following relations in $J(K)$:

$$\sum_{i=1}^{2g+1} P_i = 0, \quad P_i' = -P_i.$$

Theorem 3.1. *With the above notation, assume that* (12) *has no multiple roots, i.e. u_i are mutually distinct. Then the Mordell-Weil group $J(K)$ is a torsion-free abelian group of rank $r = 2g$. More precisely, the narrow MWL $J(K)^0$ is isomorphic to the root lattice A_{2g} and the full MWL $J(K)$ is isomorphic to its dual lattice A_{2g}^*:*

(14)
$$\begin{array}{ccccl} J(K) & \simeq & A_{2g}^* & & \\ \cup & & \cup & & \textit{index } 2g+1. \\ J(K)^0 & \simeq & A_{2g} & & \end{array}$$

Moreover $\{P_i, P_i'\}$ correspond to the minimal vectors of A_{2g}^* (with minimal norm $2g/2g+1$); in particular, $\{P_1, \ldots, P_{2g}\}$ forms a set of free generators of $J(K)$.

The algebraic surface S associated with (11) has a unique reducible fibre at $t = \infty$, and we have

$$\mathrm{rk}\ T_\infty = 2g + 4, \quad \det T_\infty = 2g + 1.$$

The case $g = 1$ reduces to the known result (Case (A_2) in [6]); in that case, we have $T_\infty^- \simeq E_6$. For $g > 1$, T_∞^- is not a root lattice, and its dual graph is as follows: first take the Dynkin graph of type D_{2g+2} and then adjoin a new vertex with norm $g+1$ to each extreme vertex of the two short branches in

D_{2g+2}. The sections (P_i) (or (P_i')) pass through the irreducible component corresponding to one (or the other) of the two new vertices.

Example. For any g, consider the curve of genus g over $\boldsymbol{Q}(t)$ defined by

$$y^2 = x(x^2-1)(x^2-2^2)\cdots(x^2-g^2)+t^2.$$

Then the Mordell-Weil group $J(\boldsymbol{Q}(t))$ of the Jacobian variety J is a free abelian group of rank $2g$ generated by

$$(x, y) = (\pm 1, t), (\pm 2, t), \ldots, (\pm g, t) \in \Gamma(\boldsymbol{Q}(t)).$$

By a standard argument, specializing t to rational numbers yields an infinite family of g-dimensional Jacobian varieties over $\boldsymbol{Q}$ of rank at least $2g$.

Finally we note that the above family $\{\Gamma_\lambda\}$ forms an excellent family of genus g curves with Galois group $W(A_{2g}) \simeq \boldsymbol{S}_{2g+1}$ in the sense of [7], § 1. It will be evident that the Galois representation

$$\rho_\lambda : \mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q}) \to \mathrm{Aut}(J(\bar{\boldsymbol{Q}}(t)))$$

has the image $W(A_{2g})$ for most choice of $\lambda \in \boldsymbol{Q}^{2g}$.

Acknowledgements. I would like to thank K. Oguiso for his help in determining the singular fibres in § 3.

References

[1] Kodaira, K.: On compact analytic surfaces. II-III. Ann. of Math., **77**, 563–626 (1963); **78**, 1–40 (1963); Collected Works. III. Iwamami and Princeton Univ. Press, pp.1269–1372 (1975).

[2] Lang, S.: Fundamentals of Diophantine Geometry. Springer-Verlag (1983).

[3] Néron, A.: Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. Publ. Math. IHES, **21** (1964).

[4] Shioda, T.: Mordell-Weil lattices and Galois representation. I, II, III. Proc. Japan Acad., **65**A, 269–271; 296–299; 300–303 (1989).

[5] ——: On the Mordell-Weil lattices. Comment. Math. Univ. St. Pauli, **39**, 211–240 (1990).

[6] ——: Construction of elliptic curves with high rank via the invariants of the Weyl groups. J. Math. Soc. Japan, **43**, 673–719 (1991).

[7] Shioda, T., and Usui, H.: Fundamental invariants of Weyl groups and excellent families of elliptic curves (to appear).

[8] Tate, J.: On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. Sém. Bourbaki, n°306 (1965/66).

[9] ——: Algorithm for determining the type of a singular fiber in an elliptic pencil. SLN, **476**, 33–52 (1975).