39. Contiguity Relations for q-hypergeometric Function and Related Quantum Group

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§1. **Introduction.** Gauss' hypergeometric function $F = F(\alpha, \beta, \gamma ; x)$ is defined to be the power series

(1.1) $$F = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma)_n(1)_n} x^n,$$

where $(\alpha)_n = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1)$ etc. It satisfies a well-known differential equation

(1.2) $$x(1-x)\frac{d^2F}{dx^2} - \{(\alpha+\beta+1)x - \gamma\}\frac{dF}{dx} - \alpha\beta F = 0.$$

It is sometimes more convenient to write it in the form

(1.3) $$(x^{-1}\theta(\theta+\gamma-1) - (\theta+\alpha)(\theta+\beta))F = 0,$$

where $\theta = xd/dx$. Heine defined its q-analogue $\varphi = \varphi(\alpha, \beta, \gamma ; q ; x)$ by the following power series (see, for example, [3]):

(1.4) $$\varphi = \sum_{n=0}^{\infty} \frac{[\alpha]_q[\alpha+1]_q\cdots[\alpha+n-1]_q[\beta]_q[\beta+1]_q\cdots[\beta+n-1]_q}{[\gamma]_q[\gamma+1]_q\cdots[\gamma+n-1]_q[1]_q[2]_q\cdots[n]_q} x^n.$$

Here $[\alpha]_q$ denotes the basic number. That is

(1.5) $$[A]_q = \frac{q^A - q^{-A}}{q - q^{-1}}$$

for any number A.

Remark. Heine and some authors define the basic number by the formula

(1.6) $$[A]_q = \frac{1-q^A}{1-q}.$$

Some formulas look differently in this case (compare [1]). Throughout this paper, we stick to (1.5).

We introduce a shift operator T by

(1.7) $$Tf(x) = f(qx),$$

and a q-difference operator

$$[\theta]_q = \frac{T - T^{-1}}{q - q^{-1}}.$$

The latter is a q-analogue of xd/dx. We also introduce

(1.8) $$[\theta+\alpha]_q = \frac{q^\alpha T - q^{-\alpha}T^{-1}}{q - q^{-1}},$$

so that, we have

(1.9) $$[\theta+\alpha]_q x^n = [n+\alpha]_q x^n.$$

The power series (1.4) satisfies the following q-difference equation

(1.10) $$(x^{-1}[\theta]_q[\theta+\gamma-1]_q - [\theta+\alpha]_q[\theta+\beta]_q)\varphi = 0.$$

This is an analogue of (1.3).

If we fix a set of parameters (α, β, γ), then $F\ (\alpha', \beta', \gamma'; x)$ is called a contiguous function of $F(\alpha, \beta, \gamma; x)$, provided that $|\alpha-\alpha'|, |\beta-\beta'|, |\gamma-\gamma'|$ are all less than or equal to 1. It is known that there are differential operators of order 1 which produce contiguous functions out of $F(\alpha, \beta, \gamma; x)$ ([9,7]). These operators can be labeled as E_{ij} $(1 \leq i, j \leq 4, i \neq j)$, and they correspond to basis elements of the Lie algebra $\mathfrak{sl}(4)$. We introduce a new set of parameters λ_i, $i=1, 2, 3, 4$ by the following relations

$$\alpha=\lambda_2, \quad \beta=1-\lambda_4, \quad \gamma=\lambda_2+\lambda_3=2-\lambda_1-\lambda_4,$$

with $\sum_{i=1}^{4}\lambda_i=2$. Then E_{ij} increases λ_i and decreases λ_j by 1. For example, E_{21} raises α and γ, while E_{13} simply lowers γ. This new set of parameters stems from the Grassmann point of view of Gelfand *et al* ([4,5,6]) that F should be regarded as a function on the Grassmannian $G_{2,4}$ of the 2-planes in a 4-space, on which SL(4) acts on the right.

In this paper, we explicitly write down the contiguity relations for Heine's series in terms of q-difference operators and the shift operator. It turns out that these operators constitute a representation of $U_q(\mathrm{SL}(4))$, the q-analogue of the universal enveloping algebra of the Lie group SL(4) (see[2,8]).

§2. Contiguity operators. We first introduce the following 4 obvious operators acting on Heine's series.

(2.1) $$E_{23}=-[\theta+\alpha]_q,$$

(2.2) $$E_{14}=-[\theta+\beta]_q,$$

(2.3) $$E_{13}=[\theta+\gamma-1]_q,$$

(2.4) $$E_{24}=x^{-1}[\theta]_q.$$

To describe the operation of E_{ij} on Heine's series, we introduce the following notation. We simply write φ instead of $\varphi(\alpha, \beta, \gamma; q; x)$ and indicate the contiguous functions by super and subscripts. For example

$$\varphi^{\alpha}=\varphi(\alpha+1, \beta, \gamma; q; x),$$
$$\varphi_{\gamma}=\varphi(\alpha, \beta, \gamma-1; q; x),$$
$$\varphi^{\alpha\beta\gamma}=\varphi(\alpha+1, \beta+1, \gamma+1; q; x).$$

The above 4 operators satisfy

(2.5) $$E_{23}\varphi=-[a]_q\varphi^{\alpha},$$

(2.6) $$E_{14}\varphi=-[\beta]_q\varphi^{\beta},$$

(2.7) $$E_{13}\varphi=[\gamma-1]_q\varphi_{\gamma},$$

(2.8) $$E_{24}\varphi=\frac{[a]_q[\beta]_q}{[\gamma]_q}\varphi^{\alpha\beta\gamma}.$$

We note (1.10) is written in the form

$$(E_{24}E_{13}-E_{14}E_{23})\varphi=0.$$

By analogy of the classical factorization method, we obtain

(2.9) $$E_{32}=x[\theta+\beta]_q-[\theta+\gamma-\alpha]_q, \qquad E_{32}\varphi=[\alpha-\gamma]_q\varphi_{\alpha},$$

(2.10) $$E_{41}=x[\theta+\alpha]_q-[\theta+\gamma-\beta]_q, \qquad E_{41}\varphi=[\beta-\gamma]_q\varphi_{\beta},$$

(2.11) $$E_{31}=[\theta+\alpha+\beta-\gamma]_q-x^{-1}[\theta]_q, \qquad E_{31}\varphi=\frac{[\alpha-\gamma]_q[\beta-\gamma]_q}{[\gamma]_q}\varphi^{\gamma},$$

(2.12) $$E_{42}=x[\theta+\alpha+\beta-1]_q-[\theta+\gamma-1]_q, \qquad E_{42}\varphi=[1-\gamma]_q\varphi_{\alpha\beta\gamma}.$$

We further define

(2.13) $E_{12}=(q^{\alpha}x[\theta+\beta]_q-[\theta+\gamma-1]_q)T,\qquad E_{12}\varphi=[1-\gamma]_q\varphi_{\alpha\gamma},$

(2.14) $E_{21}=(q^{\beta}[\theta+\alpha]_q-q^{\gamma-1}x^{-1}[\theta]_q)T,\qquad E_{21}\varphi=\dfrac{[\alpha]_q[\gamma-\beta]_q}{[\gamma]_q}\varphi^{\alpha\gamma},$

(2.15) $E_{34}=-(q^{\alpha}[\theta+\beta]_q-q^{\gamma-1}x^{-1}[\theta]_q)T,\qquad E_{34}\varphi=-\dfrac{[\beta]_q[\gamma-\alpha]_q}{[\gamma]_q}\varphi^{\beta\gamma},$

(2.16) $E_{43}=-(q^{\beta}x[\theta+\alpha]_q-[\theta+\gamma-1]_q)T,\qquad E_{43}\varphi=[1-\gamma]_q\varphi_{\beta\gamma}.$

We remark that these are determined by the following relations:

(2.17) $E_{13}E_{32}-q^{-1}E_{32}E_{13}=q^{\lambda_3-1}E_{12},$

(2.18) $E_{14}E_{42}-qE_{42}E_{14}=q^{1-\lambda_4}E_{12},$

(2.19) $E_{23}E_{31}-qE_{31}E_{23}=q^{1-\lambda_3}E_{21},$

(2.20) $E_{24}E_{41}-q^{-1}E_{41}E_{24}=q^{\lambda_4-1}E_{21},$

(2.21) $E_{42}E_{23}-q^{-1}E_{23}E_{42}=q^{\lambda_2-1}E_{43},$

(2.22) $E_{41}E_{13}-qE_{13}E_{41}=q^{1-\lambda_1}E_{43},$

(2.23) $E_{32}E_{24}-qE_{24}E_{32}=q^{1-\lambda_2}E_{34},$

(2.24) $E_{31}E_{14}-q^{-1}E_{14}E_{31}=q^{\lambda_1-1}E_{34}.$

Theorem 1. *These 12 operators E_{ij}, $i\neq j$ give contiguity relations which, in the limit $q\to 1$, reduce to those for Gauss' hypergeometric function.*

§3. Representaion of $U_q(\mathrm{SL}(4))$. We set

$$e_i=E_{i,i+1},\qquad f_i=E_{i+1,i},\qquad i=1,2,3.$$

By a direct calculation, the commutator $[e_i,f_i]=e_if_i-f_ie_i$ satisfies

(3.1) $[e_i,f_i]\varphi=[\lambda_i-\lambda_{i+1}]_q\varphi,\qquad i=1,2,3.$

In view of these, we define

(3.2) $q^{h_i}\varphi=q^{\lambda_i-\lambda_{i+1}}\varphi,\qquad i=1,2,3.$

Then we have

(3.3) $[e_i,f_i]\varphi=\dfrac{q^{h_i}-q^{-h_i}}{q-q^{-1}}\varphi,\qquad i=1,2,3.$

We also have

(3.4) $q^{h_i}e_jq^{-h_i}\varphi=q^{a_{ij}}e_j\varphi,$

(3.5) $q^{h_i}f_jq^{-h_i}\varphi=q^{-a_{ij}}f_j\varphi,$

where $a_{ii}=2$, $a_{i,i\pm1}=-1$, and $a_{ij}=0$ for the rest. In order to show that these operations give a representaion of the q-analogue $U_q(\mathrm{SL}(4))$ of the universal enveloping algebra of SL(4), we check that the following equalities hold in addition to (3.3) - (3.5) (see[8]).

(3.6) $e_i^2e_{i\pm1}-(q+q^{-1})e_ie_{i\pm1}e_i+e_{i\pm1}e_i^2=0,$

(3.7) $f_i^2f_{i\pm1}-(q+q^{-1})f_if_{i\pm1}f_i+f_{i\pm1}f_i^2=0,$

(3.8) $e_ie_j=e_je_i\quad\text{for } |i-j|>1,$

(3.9) $f_if_j=f_jf_i\quad\text{for } |i-j|>1,$

(3.10) $e_if_j=f_je_i\quad\text{for } i\neq j.$

(The double signs are to be read in same order in (3.6) (3.7).)

We can state our main result as follows.

Theorem 2. *These actions of $\{e_i, f_i, q^{h_i}\}$ determine a representation of $U_q(\mathrm{SL}(4))$.*

This is valid not only for (1.4), but also for any solution of (1.10). In a forthcoming paper, we shall study a similar problem for a q-analogue of Lauricella's F_D of l variables, which is related to the Grassmannian $G_{2,l+3}$, and therefore to $SL(l+3)$.

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