25. On Certain Real Quadratic Fields with Class Number 2

By Shigeru KATAYAMA
College of General Education, Tokushima Bunri University

(Communicated by Shokichi IYANAGA, M. J. A., April 12, 1991)

Let D be a square-free rational integer and $\varepsilon_D=(t+u\sqrt{D})/2$ $(t, u>0)$ be the fundamental unit of $\boldsymbol{Q}(\sqrt{D})$ with $N\varepsilon_D=-1$, where N is the norm map from $\boldsymbol{Q}(\sqrt{D})$ to $\boldsymbol{Q}$. Then D is expressed in the form $D=u^2n^2\pm 2an+b$, where n, a and b are integers such that $n\geqq 0$, $0\leqq a<u^2/2$ and $a^2+4=bu^2$ (cf. [6]). We denote by $h(D)$ the class number of $\boldsymbol{Q}(\sqrt{D})$. In our previous paper [1], we treated the problem of enumerating the real quadratic fields $\boldsymbol{Q}(\sqrt{D})$ with $h(D)=1$ and $1\leqq u\leqq 300$ (the cases $u=1$ and $u=2$ were treated in [3]).

In this paper, we shall consider the same problem for real quadratic fields $\boldsymbol{Q}(\sqrt{D})$ with $h(D)=2$ and $1\leqq u\leqq 200$.

We note here that the list in [4] is incomplete as it misses $\boldsymbol{Q}(\sqrt{3365})$ whereas $h(3365)=2$.

In the same way as in [1], we have the following theorem.

Theorem. *With the notation as above, there exist* 45 *real quadratic fields* $\boldsymbol{Q}(\sqrt{D})$ *with class number two for* $1\leqq u\leqq 200$, *where* D *are those in table with one possible exception.*

Proof. Let d be the discriminant of $\boldsymbol{Q}(\sqrt{D})$, that is, $d=D$ or $4D$, according as $D\equiv 1 \pmod 4$ or not. Let χ_d be the Kronecker character belonging to $\boldsymbol{Q}(\sqrt{D})$ with the discriminant d and $L(s,\chi_d)$ be the corresponding L-series. Then by Theorem 2 of [5], we have for any $y\geqq 11.2$ satisfying $e^y\leqq d$

$$L(1,\chi_d)>\frac{0.655}{y}d^{-1/y}$$

with one possible exception of d.

Hence from class-number formula, we have

$$h(D)=\frac{\sqrt{d}}{2\log\varepsilon_D}L(1,\chi_d)>\frac{0.655}{y}\,\frac{\sqrt{d}\,d^{-1/y}}{2\log(u\sqrt{d})}$$
$$\geqq\frac{0.655\,e^{(y/2)-1}}{y(y+2\log u)}.$$

Put for convenience

$$g(\log u, y)=\frac{0.655\,e^{(y/2)-1}}{y(y+2\log u)}.$$

Then $g(\log u, y)$ is a monotone increasing function for $y\geqq 11.2$. Therefore for any fixed u, there exists a real number $c=c(u)$ such that $c\geqq 11.2$

and $g(\log u, c)>2$. We can take $15.1 \leqq c(u) \leqq 16.5$ for $1 \leqq u \leqq 200$. On the other hand, by the genus theory of quadratic fields, $h(D)=2$ implies $D=p_1p_2$, where p_1, p_2 are both prime such that $p_1<p_2$.

Further, let q be the least prime q such that $(D/q)=1$. Then it is known that $h(D) \geqq (\log n)/(\log q)$ (cf. [6]). Therefore if $h(D)=2$, then $q^2 \geqq n$ holds.

Hence we searched for the integers $D=u^2n^2 \pm 2an+b$ such that $D \leqq e^{c(u)}$ and $D=p_1p_2$ and $q^2 \geq n$, and calculated the class number of $\boldsymbol{Q}(\sqrt{D})$ by the help of a computer. Q.E.D.

Details of the proof and the tables of u, D, n, q, and $h(D)$ will be published elsewhere.

Acknowledgement. The author would like to express his thanks to Dr. Shin-ichi Katayama for useful suggestions.

Table

(u,	D)	(u,	D)	(u,	D)	(u,	D)	(u,	D)
(1,	85)	(2,	10)	(5,	493)	(13,	565)	(26,	58)
(1,	365)	(2,	26)	(5,	1037)	(13,	6437)	(26,	2173)
(1,	533)	(2,	65)	(5,	1781)	(17,	2165)	(26,	3293)
(1,	629)	(2,	122)	(5,	2285)	(17,	3077)	(29,	685)
(1,	965)	(2,	362)	(5,	3869)	(17,	6485)	(34,	218)
(1,	1685)	(2,	485)	(5,	5213)	(25,	1565)	(50,	314)
(1,	1853)	(2,	1157)	(10,	74)	(25,	3653)	(53,	1165)
(1,	2813)	(2,	2117)	(10,	185)	(25,	8021)	(53,	5165)
		(2,	3365)	(10,	458)			(73,	8885)
				(10,	5837)			(101,	12365)

References

[1] S.-I. Katayama and S.-G. Katayama: A note on the problem of Yokoi. Proc. Japan Acad., **67**A, 26–28 (1991).

[2] Y. Kida: UBASIC86. Nihonhyoronsha, Tokyo (1989).

[3] H. K. Kim, M.-G. Leu and T. Ono: On two conjectures on real quadratic fields. Proc. Japan Acad., **63**A, 222–224 (1987).

[4] M.-G. Leu: On a determination of certain real quadratic fields of class number two. J. Number Theory, **33**, 101–106 (1989).

[5] T. Tatuzawa: On a theorem of Siegel. Japanese J. Math., **21**, 163–178 (1951).

[6] H. Yokoi: Some relations among new invariants of prime number p congruent to 1 mod 4. Adv. Studies in Pure Math., **13**, 493–501 (1988).