24. Spherical Functions on Some p-adic Classical Groups

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Introduction. We write down explicitly the so called Satake transform of Hecke algebra of some $\mathfrak{p}$-adic classical groups (O) with $n\neq 2\nu$, (Sp), (U), (U^{+}), (U^{-}), using Macdonald's idea for the $\mathfrak{p}$-adic Chevalley groups ([2]). For our purpose, we have only to evaluate zonal spherical functions (in §2) and the number of double cosets by the maximal compact subgroup K (in §3). The details containing the case (O) with $n=2\nu$ will be published elsewhere.

§1. **Preliminaries.** Let k be a $\mathfrak{p}$-adic field where $\mathfrak{p}$ does not lie over 2. Let k' be either k itself, a quadratic extension of k or the (unique) central division quaternion algebra over k. $\mathcal{O}$ denotes the maximal order of k'. We denote by e the ramification index of k'/k, and $\mathcal{P}=(\Pi)$ (resp. $\mathfrak{p}=(\pi)$) the prime ideal in k' (resp. k). We denote by $x\to\bar{x}$ $(x\in k')$ the canonical involution. Let ε be an element of the center of k' such that $\varepsilon\bar{\varepsilon}=1$, V a right vector space over k' of dimension n, and $\langle\ ,\ \rangle$ a non-degenerate ε-hermitian form on V, i.e., a k-bilinear mapping $V\times V\to k'$ such that

$$\langle x, y\rangle=\varepsilon\overline{\langle y, x\rangle},\quad \langle xa, yb\rangle=\bar{a}\langle x, y\rangle b \quad \text{foy all } x, y\in V,\ a, b\in k'.$$

It is known that we have the following five cases.

(O) $k'=k$ and $\varepsilon=1$.

(Sp) $k'=k$ and $\varepsilon=-1$.

(U) k' is a quadratic extension of k and $\varepsilon=1$.

(U^{+}) k' is a division quaternion algebra over k and $\varepsilon=1$.

(U^{-}) k' is a division quaternion algebra over k and $\varepsilon=-1$.

Now, let ν be the Witt index of V and put $n=n_0+2\nu$. There exists a (not uniquely determined) system of vectors $\{e_i, e_i'\ (1\leq i\leq\nu)\}$ such that

$$\langle e_i, e_j\rangle=\langle e_i', e_j'\rangle=0,\quad \langle e_i, e_j'\rangle=\delta_{ij} \quad \text{for all } i, j,$$

(δ_{ij} is Kronecker's symbol). Set

$$V_0=(\Sigma e_ik'+\Sigma e_i'k')^{\perp},\quad L_0=\{x\in V_0\,|\,\langle x, x\rangle\in\mathcal{O}\},\quad L=\Sigma e_i\mathcal{O}+\Sigma e_i'\mathcal{O}+L_0.$$

Then L is a maximal lattice and there is a system of vectors $\{f_i\ (1\leq i\leq n_0)\}$ such that

$$L_0=\Sigma f_i\mathcal{O},\quad \langle f_i, f_j\rangle=0,\quad \text{if } i\neq j.$$

We define α (resp. β) to be the number of $\{f_i\}$ such that $\langle f_i, f_i\rangle\in\mathcal{O}^{\times}$ (resp. $\langle f_i, f_i\rangle\in\mathcal{P}$). Note that $\alpha+\beta=n_0$.

We now take this basis $\{e_1, \cdots, e_\nu, f_1, \cdots, f_{n_0}, e_\nu', \cdots, e_1'\}$ and identify a k'-linear transformation g of V with a matrix (g_{ij}) by

$$g: (e_1, \cdots, e_1')\to(e_1, \cdots, e_1')(g_{ij}).$$

Let G be the connected component of the group of similitudes of V, that is

$$\tilde{G}=\{g\in GL(V)\,|\,\langle gx, gy\rangle=\mu(g)\langle x, y\rangle \text{ for all } x, y\in V \text{ and } \mu(g)\in k^{\times}\};$$

here we call $\mu(g)$ the multiplier of g.

As in [3] we set

$$K=\{k\in G\,|\,kL=L\}=\{k\in G\,|\,k, k^{-1}\in GL(n, \mathcal{O})\},$$

let G_0 be the group of similitudes of V_0, μ_0 the multiplier function of G_0 and lastly

$$N=\left\{n=\begin{bmatrix} A & 0 & 0 \\ * & {}^{I}n_0 & 0 \\ * & * & B \end{bmatrix}\in G\right\},$$

where $A, B(\in GL(\nu, k'))$ are lower triangular matrix with their diagonal elements equal to 1. Notice that K is a maximal compact subgroup corresponding to the maximal lattice L. We define the symbol [2] and e_0 by

$$[2]=\begin{cases}1 & \text{if } n_0=0,\\ 2 & \text{if } n_0>0,\end{cases}\quad \mathrm{ord}_p\mu_0(G_0)=\frac{[2]}{e_0}\boldsymbol{Z}.$$

Now we set $M=(1/e)\boldsymbol{Z}^{\nu}\times(1/e_0)\boldsymbol{Z}$, and for $(m)=(m_1/e, \cdots, m_\nu/e, m_0/e_0)\in M$, we define $\Pi^{(m)}$ as follows:

$$\Pi^{(m)}=\begin{cases}\mathrm{diag}(\Pi^{m_1}, \cdots, \Pi^{m_\nu}, \pi^{m_0}\bar{\Pi}^{-m_\nu}, \cdots, \pi^{m_0}\bar{\Pi}^{-m_1}) & \text{if } n_0=0,\\ \mathrm{diag}(\Pi^{m_1}, \cdots, \Pi^{m_\nu}, w^{m_0}, \mu_0(w)^{m_0}\bar{\Pi}^{-m_\nu}, \cdots, \mu_0(w)^{m_0}\bar{\Pi}^{-m_1}) & \text{if } n_0>0,\end{cases}$$

where w denotes an arbitralily fixed element of G_0 such that $\mathrm{ord}_p\mu_0(w)=2/e_0$. We denote by D a subgroup in G generated by $\Pi^{(m)}$ with $(m)\in M$. It is known ([3]) that

$G=KDK=KDN$ (Cartan decomposition and Iwasawa decomposition).

Now we can define zonal spherical functions on G relative to K. For $x\in G$, it is given by

$$\omega_s(x^{-1})=\int_K \phi_s(xk)dk,$$

where $\phi_s(x)=s(d)\delta^{1/2}(d)$ for $x=kdn$ (Iwasawa decomposition), s is a character of the group D, δ is a function of D defined by $d(dnd^{-1})/d(n)=\delta^{-1}(d)$, and the Haar measure dk is normalized by

$$\int_K dk=1.$$

As $\omega_s(x)$ is constant on each coset KxK, we will henceforce suppose $x\in D$. We use some notions of Lie algebras i.e., root system and of Weyl group.

In a ν-dimentional vector space over $\boldsymbol{R}$ with the standard basis $\varepsilon_1, \cdots, \varepsilon_\nu$, the root system Σ_0 (resp. positive root Σ^+) is given as follows:

$$\Sigma_0=\begin{cases}(C_\nu)=\{\pm 2\varepsilon_i, \pm\varepsilon_i\pm\varepsilon_j\,|\,1\leq i, j\leq\nu, i\neq j\} & \text{for } (Sp),\\ (B_\nu)=\{\pm\varepsilon_i, \pm\varepsilon_i\pm\varepsilon_j\,|\,1\leq i, j\leq\nu, i\neq j\} & \text{for } (O),\\ (BC_\nu)=\{\pm\varepsilon_i, \pm 2\varepsilon_i, \pm\varepsilon_i\varepsilon_j\,|\,1\leq i, j\leq\nu, i\neq j\} & \text{for } (U), (U^+), (U^-),\end{cases}$$

$$\left(\text{resp. } \Sigma^+=\begin{cases}(C_\nu)=\{2\varepsilon_i, \varepsilon_j+\varepsilon_k, \varepsilon_j-\varepsilon_k\,|\,1\leq i\leq\nu, 1\leq j<k\leq\nu\}\\ (B_\nu)=\{\varepsilon_i, \varepsilon_j+\varepsilon_k, \varepsilon_j-\varepsilon_k\,|\,1\leq i\leq\nu, 1\leq j<k\leq\nu\}\\ (BC_\nu)=\{\varepsilon_i, 2\varepsilon_i, \varepsilon_j+\varepsilon_k, \varepsilon_j-\varepsilon_k\,|\,1\leq i\leq\nu, 1\leq j<k\leq\nu\}\end{cases}\right)$$

Moreover the simple roots of the root systems are $\{2\varepsilon_\nu, \varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq\nu-1)\}$ for (Sp), $\{\varepsilon_\nu, \varepsilon_i-\varepsilon_{i+1}\ (1\leq i\leq\nu-1)\}$ for (O), (U), (U^+) and (U^-). The Weyl group W (operating on M) is generated by all permutations of $(m_1, \cdots, m_\nu)$

and by the automorphisms $w^{(i)}$ $(1\leq i\leq \nu)$ defined by

$$w^{(i)}=\begin{cases} m_i\to -m_i+[2]\dfrac{e}{e_0}m_0, \\ m_j\to m_j \quad (j\neq i). \end{cases}$$

Now we can see any permutation $(m_1,\cdots,m_\nu)\to(m_{p(1)},\cdots,m_{p(\nu)})$ operates on Σ_0 by transforming ε_i to $\varepsilon_{p(i)}$ in Σ_0 for all i. Moreover for each i we can see $w^{(i)}$ operates on Σ_0 by transforming ε_i to $-\varepsilon_i$ and fixing all ε_j $(j\neq i)$.

We denote this operation of W in Σ_0 by $w(a)$ (or simply wa) for $a\in\Sigma_0$, $w\in W$. It is easy to see that there is an element w_a of W corresponding to any simple root $a\in\Sigma^+$ such that $w_a(a)=-a$ and $w_a(\Sigma^+\backslash\{a,2a\})=\Sigma^+\backslash\{a,2a\}$ (if a, $2a\in\Sigma^+$, then $w_a=w_{2a}$).

We call these w_a simple reflections. We can consider that W operates on D canonically. Now let $w=w_1\cdots w_r\in W$ be a "reduced" word where w_i's are simple reflections, that is to say, w is not a product of r' simple reflections for $r'<r$. We denote r by $l(w)$.

§ 2. **Zonal spherical functions.** For a character s of D, we define $C_0(\varepsilon_\nu,s)\cdot C_0(2\varepsilon_\nu,s)$ and $C_0(\varepsilon_i-\varepsilon_{i+1},s)$ for (U), (U^+), (U^-), and $C_0(a,s)$ for simple roots a for (O), (Sp) as follows:

i) Setting $\Pi_i=\mathrm{diag}(\underbrace{1,\cdots,1}_{i-1},\Pi,\Pi^{-1},1,\cdots,1,\bar{\Pi},\bar{\Pi}^{-1},\underbrace{1,\cdots,1}_{i-1})\in D$,

$$C_0(\varepsilon_i-\varepsilon_{i+1},s)=\frac{q-S}{q-qS},\quad \text{with } q=|\mathcal{O}/\mathcal{P}|,\ S=s(\Pi_i),\ 1\leq i\leq \nu-1.$$

ii) Setting $\Pi_\nu=\mathrm{diag}(\underbrace{1,\cdots,1}_{\nu-1},\Pi,1,\cdots,1,\bar{\Pi}^{-1},\underbrace{1,\cdots,1}_{\nu-1})\in D$,

$$C_0(2\varepsilon_\nu,s)=1+\frac{T}{1-S^2}\ \text{ for } (Sp),\quad C_0(\varepsilon_\nu,s)=1+\frac{T'}{1-S^2}\ \text{ for } (O),$$

$$C_0(\varepsilon_\nu,s)C_0(2\varepsilon_\nu,s)=1+\frac{T''}{1-S^2}\ \text{ for } (U),\ (U^+),\ (U^-),$$

with $q=|\mathcal{O}/\mathcal{P}|$, $S=s(\Pi_\nu)$, where T, T' and T'' are defined as follows:

$$T=\frac{(q^\beta-1)}{q^{(\alpha+\beta)/2}}S+\frac{q^\beta(q^\alpha-1)}{q^{(\alpha+\beta)}}S^2\ \text{ for } (O),\quad T'=(1+S)S\left(1-\frac{1}{q}\right)\ \text{ for } (Sp) \text{ and}$$

$$T''=\begin{cases} \dfrac{(q^{\beta+1/2}-1)}{q^{(\alpha+\beta+1)/2}}S+\dfrac{q^{\beta+1/2}(q^{\alpha+1/2}-1)}{q^{(\alpha+\beta+1)}}S^2 & \text{for } (U) \text{ with } e=1, \\ \dfrac{q-1}{q^{(\alpha+1)/2}}S+\dfrac{q(q^\alpha-1)}{q^{(\alpha+1)}}S^2 & \text{for } (U) \text{ with } e=2, \\ \dfrac{q-1}{q^{(\alpha+3/2)/2}}S+\dfrac{q(q^{(1/2+\alpha)}-1)}{q^{(\alpha+3/2)}}S^2 & \text{for } (U^+), \text{ and} \\ \dfrac{q^\beta-1}{q^{(1/2+\alpha+\beta)/2}}S+\dfrac{q^\beta(q^{(1/2+\alpha)}-1)}{q^{(1/2+\alpha+\beta)}}S^2 & \text{for } (U^-). \end{cases}$$

And for other roots $a\in\Sigma^+$, we define $C_0(a,s)$ by the property $C_0(a,s)=C_0(wa,ws)$ for all $w\in W$. Moreover we set $C(s)=\prod_{a\in\Sigma^+}C_0(a,s)$ and $(ws)(x)=s(w^{-1}xw)$ with $x\in D$.

The explicit formula of zonal spherical functions is given as follows:

Theorem 1. (The formula for spherical functions.) *If the denominator of the rational function $C_0(a, s)$ in q does not vanish, then the spherical function is given as follows:*

$$\omega_s(x^{-1}) = \kappa \cdot \delta^{1/2}(x) \sum_{w \in W} C(ws^{-1}) \cdot (ws)(x).$$

for

$$x = \Pi^{(m)}, \quad (m) = \left(\frac{m_1}{e}, \cdots, \frac{m_\nu}{e}, \frac{m_0}{e_0}\right) \quad \textit{with } m_1 \geq \cdots \geq m_\nu \geq \frac{[2]e}{2e_0} m_0.$$

Here κ is some constant which is evaluated by substituting $s = \delta^{-1/2}$.

§3. Calculation of $[KxK: K]$ for $x \in D$. We can write x as

$$x = \Pi^{(m)}, \quad (m) = \left(\frac{m_1}{e}, \cdots, \frac{m_\nu}{e}, \frac{m_0}{e}\right) \quad \text{with } m_1 \geq \cdots \geq m_\nu \geq \frac{[2]e}{2e_0} m_0.$$

Let W_x be the subgroup of W consisting of all w such that $w^{-1}xw = x$, and define $l'(w)$ to be the number of w_{ε_ν} ($= w_{2\varepsilon_\nu}$ if $2\varepsilon_\nu \in \Sigma^+$) which appears in the shortest expression of w.

Set $L(w) = l(w) + (\eta - 1)l'(w)$ for $w \in W$, where $\eta = 1$ for (Sp), $= \alpha$ for (O) and (U) with $e = 2$, $= \alpha + 1/2$ for other cases. We have

Theorem 2. *For x in D,*

$$[KxK: K] = \delta(x) \cdot \sum_{w \in W} q^{-L(w)} \cdot \left(\sum_{w \in W_x} q^{L(w)}\right).$$

References

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