72. An Additive Problem of Prime Numbers. III

By Akio FUJII
Department of Mathematics, Rikkyo University

(Communicated by Shokichi IYANAGA, M. J. A., Oct. 14, 1991)

§1. Let γ run over the imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$. We assume the Riemann Hypothesis throughout this article. Here we are concerned with the value distribution of the bounded oscillating quantity $G(X)$ for $X\geq 1$ defined by

$$G(X)\equiv\Re\left\{\sum_{\gamma>0}\frac{X^{i\gamma}}{(1/2+i\gamma)(3/2+i\gamma)}\right\}.$$

This function plays important roles in some problems in the analytic theory of numbers. We may recall two formulas involving $G(X)$. One is concerned with Goldbach's problem on average and the other is concerned with the prime number theorem on average.

(I) *For $X>X_0$, we have*

$$\sum_{n\leq X}\left\{\sum_{m+k=n}\Lambda(m)\Lambda(k)-n\cdot\prod_{p|n}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid n}\left(1-\frac{1}{(p-1)^2}\right)\right\}$$
$$=-4X^{3/2}G(X)+O((X\log X)^{1+1/3}),$$

where $\Lambda(n)$ is the von Mangoldt function.

(II) *For $X\geq 1$, we have*

$$\int_0^X\left(\sum_{n\leq y}\Lambda(n)-y\right)dy=-2X^{3/2}G(X)-X\log(2\pi)+\log(2\pi)+C_0$$
$$-1-(6/\pi^2)\zeta'(2)-X\sum_{a=1}^{\infty}(X^{-2a}/2a(2a-1)),$$

where C_0 is the Euler constant.

(I) has been proved in the author's previous work [7]. (II) is known to hold without assuming any unproved hypothesis in the following form (cf. p. 52 and p. 74 of Edwards [5]). For $X\geq 1$,

$$\int_0^X\left(\sum_{n\leq y}\Lambda(n)-y\right)dy=-\sum_{\substack{\zeta(\rho)=0\\0<\Re(\rho)<1}}\frac{X^{\rho+1}}{\rho(\rho+1)}-X\sum_{a=1}^{\infty}\frac{X^{-2a}}{2a(2a-1)}-\frac{\zeta'}{\zeta}(0)X+\frac{\zeta'}{\zeta}(-1).$$

In (II), $G(X)$ is the only oscillating part. However in (I), the remainder term has still another oscillating property connected with the distribution of the zeros of $\zeta(s)$ as has been seen in [6] and [7].

We notice that the formula (II) implies, for example, that

$$G(1)=(1/2)(-(1/2)+C_0-(6/\pi^2)\zeta'(2)-\log 2)$$

and

$$G(2)=(1/4\sqrt{2})(1-\log\pi+C_0-(6/\pi^2)\zeta'(2)+\log 2-(3/2)\log 3).$$

Generally, we have for $X>1$,

$$G(X)+(1/2X^{3/2})\{(X-1)\log\pi-C_0+(6/\pi^2)\zeta'(2)\}-(1/2X^{3/2})\{(X^2/2)-1\}$$
$$=-(1/2X^{3/2})\{(X-1)\log 2+\log A_1+(X-[X])\log A_2$$
$$-\log(1-(1/X))+((X+1)/2)\log(1-(1/X^2))\},$$

where A_1 and A_2 are the integers defined by

$$A_1=\begin{cases}\prod_{2\le n\le [X]-1}\prod_{p\le n} p^{[\log n/\log p]} & \text{if } X\ge 3\\ 1 & \text{if } 1<X<3\end{cases}$$

and

$$A_2=\begin{cases}\prod_{p\le [X]} p^{[\log [X]/\log p]} & \text{if } X\ge 2\\ 1 & \text{if } 1<X<2,\end{cases}$$

$[X]$ being the Gauss symbol. Since the right hand side is $\neq 0$ (in fact, it is <0), we get the following consequence by applying Baker's Theorem 2 in [2] on the linear combination of the logarithms of the algebraic numbers.

Corollary 1. *If $X(\ge 1)$ is an algebraic number, then*

$$G(X)+(1/2X^{3/2})\{(X-1)\log\pi-C_0+(6/\pi^2)\zeta'(2)\}$$

is a transcendental number.

Without assuming any unproved hypothesis, we see that if $X>1$ is an algebraic number, then

$$\sum_{\substack{\zeta(\rho)=0\\ 0<\Re(\rho)<1}}(X^{\rho+1}/\rho(\rho+1))+(X-1)\log\pi-C_0+(6/\pi^2)\zeta'(2)$$

is a transcendental number. More generally, we can formulate a similar result for the sum

$$\sum_{\substack{\zeta(\rho)=0\\ 0<\Re(\rho)<1}}(X^{\rho+k}/\rho(\rho+1)(\rho+2)\cdots(\rho+k))\quad\text{for } k\ge 2.$$

§ 2. We are next concerned with the value distribution of $G(X)$ as $X\to\infty$. We shall give first a rough estimate of $G(X)$. It implies, in principle, that

$$G(X)>0.012\qquad\text{for infinitely many } X$$

and

$$G(X)<-0.012\qquad\text{for infinitely many } X.$$

This should be compared with Littlewood [11].

To see this we rewrite $G(X)$ as follows.

$$G(X)=-\sum_{\gamma>0}\frac{\cos(\gamma\log X)}{\gamma^2+1/4}+\sum_{\gamma>0}\frac{3\cos(\gamma\log X)+2\gamma\sin(\gamma\log X)}{(\gamma^2+1/4)(\gamma^2+9/4)}$$
$$=-G_1(X)+G_2(X),\ \text{say}.$$

$$|G_2(X)|\le 2\sum_{\gamma>0}\frac{1}{(\gamma^2+1/4)\sqrt{\gamma^2+9/4}}$$
$$\le 2\sum_{m=1}^{3}\frac{1}{(\gamma_m^2+1/4)\sqrt{\gamma_m^2+9/4}}+\frac{2}{\sqrt{\gamma_4^2+9/4}}\left\{\sum_{m=1}^{\infty}\frac{1}{\gamma_m^2+1/4}-\sum_{m=1}^{3}\frac{1}{\gamma_m^2+1/4}\right\}$$
$$\le 0.00105+0.00094\le 0.0020,$$

where γ_m is the m-th positive imaginary part of the zeros of $\zeta(s)$ for $m=1,2,3,\cdots$ and we notice that $\gamma_1=14.1347251\cdots$, $\gamma_2=21.0220396\cdots$, $\gamma_3=25.0108575\cdots$, $\gamma_4=30.4248761\cdots$ and $\sum_{m=1}^{\infty}1/(\gamma_m^2+1/4)=0.02309\cdots$. Suppose that M is the largest integer such that $\gamma_M\le(23/25)(\gamma_1^2+1/4)$. M can be taken to be 70. Then we have

$$\left|G_1(X)-\sum_{m=1}^{M}\frac{\cos(\gamma_m\log X)}{\gamma_m^2+1/4}\right|\le\int_{(23/25)(\gamma_1^2+1/4)}^{\infty}\frac{1}{t^2+1/4}d\Big(\sum_{0<\gamma\le t}1\Big)$$
$$\le\frac{\log((23/25)(\gamma_1^2+1/4)e/2\pi)}{2\pi(23/25)(\gamma_1^2+1/4)}+\frac{1}{(\gamma_1^2+1/4)^2}\frac{3}{(23/25)^2}\left\{0.137\log\left(\frac{23}{25}\left(\gamma_1^2+\frac{1}{4}\right)\right)\right.$$

$$+0.443\log\log\left(\frac{23}{25}\left(\gamma_1^2+\frac{1}{4}\right)\right)+4.350\Big\}$$

$$\leq\frac{1}{\gamma_1^2+1/4}(0.75732+0.10274)\leq 0.0043,$$

where we have used Backlund's estimate (cf. p. 134 of [5]) for $T\geq 2$

$$\left|\sum_{0<\gamma\leq T}\cdot 1-\left(\frac{T}{2\pi}\log\frac{T}{2\pi}-\frac{T}{2\pi}-\frac{7}{8}\right)\right|\leq 0.137\log T+0.443\log\log \mathrm{T}+4.350.$$

Thus we get

$$G(X)=-\sum_{m=1}^{M}\frac{\cos(\gamma_m\log X)}{\gamma_m^2+1/4}+\Theta,$$

where

$$|\Theta|\leq 0.0063.$$

Now suppose that for $\varepsilon=1/10$, y^* and some integers k_m, $m=1,2,\cdots,70$ satisfy

$$|\gamma_m y^*-\pi-2\pi k_m|<\varepsilon \qquad \text{for } m=1,2,\cdots,70.$$

By Kronecker's theorem this is possible if $\gamma_1,\gamma_2,\cdots$ and γ_{70} are linearly independent over the rationals. In fact, we do not need to assume it. We could find such solutions by Odlyzko and te Riele [14], where they have encountered a similar problem and have disproved Mertens Conjecture by using them. Thus we get

$$G(e^{y^*})=\textstyle\sum_{m=1}^{M}(1/(\gamma_m^2+1/4))+\Theta'+\Theta,$$

where

$$|\Theta'|\leq\varepsilon^2\textstyle\sum_{m=1}^{M}(1/(\gamma_m^2+1/4))\leq\varepsilon^2\cdot 0.02309\cdots\leq 0.0003.$$

Since

$$|\Theta+\Theta'|\leq 0.0066 \quad\text{and}\quad \textstyle\sum_{m=1}^{70}(1/\gamma_m^2+1/4)\geq 0.0187,$$

we get y^* such that $G(e^{y^*})>0.012$.

Similarly, one could get y^{**} such that $G(e^{y^{**}})<-0.012$.

§3. Here we shall seek a more precise information concerning the value distribution of $G(X)$ as $X\to\infty$. We propose the following problem.

Problem. *To study the function $g(\beta)$ of β defined by*

$$g(\beta)=\lim_{X\to\infty}(1/X)|\{\alpha\in[1,X],G(\alpha)\leq\beta\}|,$$

assuming that it exists, where $|\beta|\leq\sum_{\gamma>0}(1/\sqrt{\gamma^2+1/4}\cdot\sqrt{\gamma^2+9/4})$.

A more general problem may be the value distribution of

$$\Psi(X)=\textstyle\sum_{\gamma>0}(X^{i\gamma}/(1/2+i\gamma)(3/2+i\gamma)).$$

This problem corresponds to the value distribution of

$$\log\zeta(\sigma_0+it)$$

for any $\sigma_0>1$. Bohr and Jessen (cf. [3], [4] and Chap. XI of [17]) constructed a beautiful theory to this, where the linear independence of the logarithms of prime numbers is essential. We shall describe below some consequences of the analogue of Bohr-Jessen's theory under the following assumption.

(A): *γ_m's are linearly independent over the rationals.*

We put

$$f(\alpha)=\Psi(e^{\alpha})=\sum_{m=1}^{\infty}r_m e^{i(\alpha\gamma_m+\Psi_m)}$$

with

$$r_m=(1/\sqrt{\gamma_m^2+1/4}\cdot\sqrt{\gamma_m^2+9/4})$$

and

$$\Psi_m=-\arg(((3/2)+i\gamma_m)((1/2)+i\gamma_m)) \qquad \text{for } m=1,2,3,\cdots.$$

We put also

$$\Phi(\theta)=\Phi(\theta_1,\theta_2,\theta_3,\cdots,\theta_m,\cdots)=\sum_{m=1}^{\infty} r_m e^{2\pi i\theta_m},$$

where $0\leq\theta_m\leq 1$.

Under (A), for any $\varepsilon>0$, there exist N, α and $\theta_1', \theta_2', \theta_3', \cdots, \theta_N'$ such that

$$\left|\sum_{m=N+1}^{\infty} r_m e^{i(\alpha\gamma_m+\Psi_m)}\right|<\varepsilon, \qquad \left|\sum_{m=N+1}^{\infty} r_m e^{2\pi i\theta_m'}\right|<\varepsilon$$

and

$$\left|\sum_{m=1}^{N} r_m e^{i(\alpha\gamma_m+\Psi_m)}-\sum_{m=1}^{N} r_m e^{2\pi i\theta_m'}\right|<\varepsilon.$$

Moreover, the situation is simpler since the sums of the convex curves in Bohr-Jessen's theory is here the sums of the circles. Thus we get the following results.

(III) {the values taken by $f(\alpha)$} *is everywhere dense in the set* {the velues taken by $\Phi(\theta)$}. *Moreover we have*

$$\{\text{the values taken by } \Phi(\theta)\}=\left\{w\,;\,|w|\leq\sum_{\gamma>0}\frac{1}{\sqrt{\gamma^2+1/4}\cdot\sqrt{\gamma^2+9/4}}\right\}.$$

(IV) *For any closed rectangle R in the complex plane, we have*

$$\lim_{X\to\infty}\frac{1}{X}|\{0\leq\alpha\leq X\,;\,f(\alpha)\in R\}|=\iint_R F(x+iy)\,dx\,dy,$$

where the continuous function $F(x+iy)$ is constructed below.

For (III), we notice only that

$$\frac{1}{\sqrt{\gamma_1^2+1/4}\cdot\sqrt{\gamma_1^2+9/4}}\leq\sum_{m=2}^{\infty}\frac{1}{\sqrt{\gamma_m^2+1/4}\cdot\sqrt{\gamma_m^2+9/4}}.$$

From (IV), we get the following consequence concerning our problem stated above.

Corollary 2. *For any β in the interval $-r\equiv-\sum_{\gamma>0}\frac{1}{\sqrt{\gamma^2+1/4}\cdot\sqrt{\gamma^2+9/4}}$ $\leq\beta\leq+r$, we have*

$$g(\beta)=\int_{-r}^{+r}\int_{-r}^{\beta}F(x+iy)\,dx\,dy.$$

We shall now describe the construction of $F(z)$. For this purpose we put

$$\Sigma_2=\{r_1e^{2\pi i\theta_1}+r_2e^{2\pi i\theta_2}\,;\,0\leq\theta_1,\theta_2\leq 1\}.$$

Let $F_2(z)$ be defined by

$$F_2(z)=\begin{cases}\dfrac{1}{\pi^2}\dfrac{1}{\sqrt{4r_1^2r_2^2-(|z|^2-r_1^2-r_2^2)^2}} & \text{if } z\in\text{“the interior of } \Sigma_2\text{”},\\ \infty & \text{if } z\in\text{“the boundary of } \Sigma_2\text{”},\\ 0 & \text{if } z\notin\Sigma_2.\end{cases}$$

Using this we define $F_N(z)$ for $N\geq 3$ by

$$F_N(z)=\int_0^1\int_0^1\cdots\int_0^1 F_2(z-r_3'e^{2\pi i\theta_3}-r_4'e^{2\pi i\theta_4}-\cdots-r_N'e^{2\pi i\theta_N})\,d\theta_3\,d\theta_4\cdots d\theta_N,$$

where we put $r_3'=r_4$, $r_4'=r_8$, $r_5'=r_3$, $r_6'=r_5$, $r_7'=r_6$, $r_8'=r_7$ and $r_n'=r_n$ for $n\geq 9$. Then we define $F(z)$ by

$$F(z)=\lim_{N\to\infty} F_N(z).$$

This $F(z)$ is the desired function as is proved in Bohr-Jessen [3] and [4] (cf. [13] for a sketch of their method).

§ 4. As a supplement to the previous sections, we may describe some remarks on the value distribution of $G_0(X)$ which is defined below and plays also an important role in the prime number theory.

$$G_0(X)=\Re\{\textstyle\sum_{\gamma>0} X^{i\gamma}/i\gamma\} \qquad \text{for } X>1.$$

$G_0(X)$ is a special value of the following zeta function $Z_\alpha(s)$ which was introduced by the author in [8] and [9] and is shown to be entire as a function of s.

$$Z_\alpha(s)=\textstyle\sum_{\gamma>0}\sin(\alpha\gamma)/\gamma^s.$$

In fact, $G_0(X)=Z_{\log X}(1)$ and it appears in the following formula (cf. Guinand [10]).

$$\begin{aligned} Z_{\log X}(1)&=-\frac{1}{2}\sum_{n\leq X}\frac{\Lambda(n)}{\sqrt{n}}+\frac{1}{4}\frac{\Lambda(X)}{\sqrt{X}}+\left(\sqrt{X}-\frac{1}{\sqrt{X}}\right)+\frac{1}{4}\log\frac{\sqrt{X}+1}{\sqrt{X}-1}\\ &\quad+\frac{1}{2}\arctan\frac{1}{\sqrt{X}}-\frac{1}{4}C_0-\frac{1}{8}\pi-\frac{1}{4}\log 8\pi\\ &=-\frac{1}{2\sqrt{X}}\{\sum_{n\leq X}\Lambda(n)-X\}+O(1). \end{aligned}$$

We notice first that for any algebraic number $X>1$,

$$\frac{\sqrt{X}+1}{\sqrt{X}-1}\neq A_3^2 P(X)^{-1/\sqrt{X}}2^3(-1)^{-\sqrt{-1}/2}e^X,$$

where we put

$$A_3=\begin{cases}\prod_{p\leq[X]} p^{\alpha(p)} & \text{if } X\geq 2\\ 1 & \text{if } 1<X<2,\end{cases}$$

$$\alpha(p)=(1-p^{(-1/2)[\log[X]/\log p]})/(\sqrt{p}-1) \quad\text{and}\quad P(X)=e^{\Lambda(X)},$$

because the right hand side is transcendental by Baker's Theorem 3 in [2]. Then using Baker's Theorem 2 in [2], we get the following consequence.

Corollary 3. *If X is an algebraic number >1, then*

$$G_0(X)-(1/2)\arctan(1/\sqrt{X})+(1/4)C_0+(1/4)\log\pi$$

is a transcendental number.

Here we may notice that the following assumption (A′) implies the existence of $X^*=e^{y^*}$ and a positive constant C_1 satisfying

$$\sum_{n\leq X^*}\Lambda(n)-X^*>C_1\sqrt{X^*}\log^2 X^*,$$

since, by pp. 255–256 of [15], for X, $T\geq 2$, we have

$$\sum_{n\leq X}\Lambda(n)-X=-2\sqrt{X}\sum_{0<\gamma\leq T}\frac{\sin(\gamma\log X)}{\gamma}+O\left(\sqrt{X}+\frac{X\log^2(XT)}{T}\right).$$

(A′): *For any $\varepsilon>0$ and any $T>T_0$, there exists a number y^* such that*

i) $e^{(1/2)y^*}A \leq T$ *for some positive constant* A

and

ii) *with some integers* m_γ,

$$|\gamma y^* + (\pi/2) - 2\pi m_\gamma| < \varepsilon \qquad \text{for } 0 < \gamma \leq T.$$

(A′) might be too strong because its consequence is much stronger than Montgomery's suggestion in the Foreward of Ingham [11].

Moreover, the value distribution of $G_0(X)$ as X varies is a little bit delicate as is seen in the following theorem proved by the author in [8] and [9].

(V) *For any prime number* p *and an integer* $k \geq 1$, *we have*

$$\lim_{m\to\infty} \sum_{0<\gamma\leq m} \frac{\sin(\gamma(\log p^k \pm \pi/m))}{\gamma} - Z_{\log p^k}(1) = \mp \frac{1}{2\pi} \frac{\log p}{p^{k/2}} \int_0^\pi \frac{\sin t}{t}\, dt$$

and

$$\lim_{\alpha\to\log p^k \pm 0} Z_\alpha(1) - Z_{\log p^k}(1) = \mp \frac{1}{2\pi} \frac{\log p}{p^{k/2}} \int_0^\infty \frac{\sin t}{t}\, dt.$$

(V) represents Gibbs's phenomenon.

References

[1] R. J. Anderson and H. M. Stark: Oscillation theorems. Lect. Notes in Math., **899**, 79–106 (1981).

[2] A. Baker: Effective methods in Diophantine problems. Proc. of Symp. in Pure Math., XX, pp. 195–205, AMS, Providence (1971).

[3] H. Bohr and B. Jessen: Über die Wertverteilung der Riemannschen Zetafunktion. Acta Math., **54**, 1–35 (1930).

[4] ——: Om Sondsynlighedsfordelinger ved Addition af konvekse Kurve. Den. Vid. Selsk. Skr. Nat. Math. Afd., (8) **12**, 1–82 (1929).

[5] H. M. Edwards: Riemann's Zeta-Function. Academic, New York, London (1974).

[6] A. Fujii: An additive problem of prime numbers. Acta Arith., LVIII, **2**, 173–179 (1991).

[7] ——: An additive problem of prime numbers. II. Proc. Japan Acad., **67**A, 148–152 (1991).

[8] ——: The zeros of the Riemann zeta function and Gibbs's phenomenon. Comment. Math. Uni. Sancti Pauli, **32**, no. 2, 229–248 (1983).

[9] ——: Zeros, eigenvalues and arithmetic. Proc. Japan Acad., **60**A, 22–25 (1984).

[10] A. P. Guinand: A summation formula in the theory of prime numbers. Proc. London Math. Soc., ser. 2, **50**, 107–119 (1945).

[11] A. E. Ingham: The distribution of prime numbers. Cambridge Mathematical Library Series, Cambridge University Press (1990).

[12] J. E. Littlewood: On a theorem concerning the distribution of prime numbers. J. London Math. Soc., **2**, 41–45 (1927).

[13] K. Matsumoto: Descrepancy estimates for the value distribution of the Riemann zeta function. I. Acta Arith., XLVII, 167–190 (1987).

[14] A. M. Odlyzko and H. J. J. te Riele: Disproof of the Mertens conjecture. Crelle J., **357**, 138–160 (1985).

[15] K. Prachar: Primzahlverteilung. Springer (1957).

[16] H. Rademacher: Remarks concerning the Riemann-von Mangoldt formula. Rep. Institute in the Theory of Numbers. Univ. of Colorado, pp. 31–37 (1959).

[17] E. C. Titchmarsh: The Theory of the Riemann Zeta Function. Oxford (1951).