70. On a Conjecture of Gackstatter and Laine on Some Differential Equations

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1. Introduction. In this paper, we consider the differential equation

$$P(z, w')=Q(z, w), \tag{1.1}$$

in the complex plane, where $P(z, w')$ and $Q(z, w)$ are polynomials of w' and w with meromorphic (maybe transcendental) coefficients, respectively:

$$\begin{cases} P(z, w')=w'^{p}+b_{p-1}(z)w'^{p-1}+\cdots+b_1(z)w' \\ Q(z, w)=a_q(z)w^q+a_{q-1}(z)w^{q-1}+\cdots+a_0(z),\ a_q(z)\not\equiv 0. \end{cases} \tag{1.1$'$}$$

We use standard notations in Nevanlinna theory [2] [5]. Let $f(z)$ be a meromorphic function. As usual, $m(r, f)$, $N(r, f)$ and $T(r, f)$ denote the proximity function, the counting function, and the characteristic function of $f(z)$, respectively.

A function $\varphi(r)$, $0\leq r<\infty$, *is said to be* $S(r, f)$ if there is a set $E\subset \mathbf{R}^+$ of finite linear measure such that $\varphi(r)=o(T(r, f))$ as $r\to\infty$, $r\notin E$. A meromorphic function $a(z)$ is *small with respect to* $f(z)$, if $T(r, a)=S(r, f)$.

Let $\Omega(z, w, w', \cdots, w^{(n)})$ be a differential polynomial of w with meromorphic coefficients and $\mathcal{M}$ be the set of its coefficients. We call a transcendental meromorphic solution $w(z)$ of the differential equation $\Omega(z, w, w', \cdots, w^{(n)})=0$ is *an admissible solution*, if $T(r, a)=S(r, w)$ for any $a(z)\in\mathcal{M}$.

Gackstatter and Laine [1] investigated the binomial equation

$$(w')^p=Q(z, w) \quad (b_{p-1}=\cdots=b_1\equiv 0 \text{ in } (1.1')) \tag{1.2}$$

and they conjectured that it would not possess any admissible solution if $1\leqq q\leqq p-1$. Some investigations have been done for this conjecture, e.g. [6] [8] [9] [10].

In [6], Ozawa pointed out that this conjecture is closely connected with a problem due to Hayman ([3] Problem 1.21). If (1.1) possesses an admissible solution $w(z)$, then from (1.1) and (1.1$'$).

$$pT(r, w')=qT(r, w)+S(r, w). \tag{1.3}$$

Thus $T(r, w)/T((r, w')\to p/q>1$ for $r\to\infty$ outside a set E of finite linear measure.

Recently, He and Laine [4] solved this conjecture affirmatively.

Theorem A. *When* $1\leqq q\leqq p-1$ *in* (1.2), *the differential equation* (1.2) *possesses no admissible solution.*

Toda [9] treated more general differential equation

$$H(z, w, w', \cdots, w^{(k)})^m=Q(z, w), \tag{1.4}$$

where $H(z, w, w', \cdots, w^{(k)})$ is a differential polynomial of w. He proved the following theorem.

Theorem B. *When* $0 \leqq q \leqq m-1$ *in* (1.4), *the differential equation* (1.4) *has no admissible solution unless it is of the following form*:

(1.5) $$H(z, w, w', \cdots, w^{(k)})^m = a_q(z)(w+\alpha(z))^q, \quad a_q(z) \not\equiv 0.$$

In this paper, we will show that Theorem A can be generalized for the equation (1.1) in place of (1.2).

Theorem 1. *When* $1 \leqq q \leqq p-1$ *in* (1.1), *the differential equation* (1.1) *possesses no admissible solution.*

2. Preliminary lemmas. We consider the equation (1.1). In the below, $\mathcal{M}$ denotes the set of the coefficients of (1.1). Let $w(z)$ be an admissible solution of (1.1) (if exists). For $c \in \boldsymbol{C} \cup \{\infty\}$, z_0 is *an admissible c-point of* $w(z)$, if $w(z_0)=c$ and if z_0 is neither zero nor pole of any functions which belong to $\mathcal{M}$.

Lemma 1. *Suppose the differential equation* (1.1) *possesses an admissible solution* $w(z)$ *for* $1 \leqq q \leqq p-1$. *Then*

(2.1) $$N(r, w) = S(r, w), \quad N(r, w') = S(r, w).$$

Proof of Lemma 1. Suppose there exists an admissible pole z_0 of $w(z)$ and let μ be its order. From (1.1), $(\mu+1)p = \mu q$, which contradicts to the condition $1 \leqq q \leqq p-1$. Hence (2.1) holds.

For the estimations of the proximity functions of some rationals of w and w', we state the following lemma.

Lemma 2. *Let* τ_j $(j=1, 2, \cdots, s)$ *be complex constants such that* $m(r, \tau_j; w) = S(r, w)$. *Then, for* $p \leqq s$

$$m\left(r, \frac{(w')^p}{\prod_{j=1}^{s}(w-\tau_j)}\right) = S(r, w).$$

The proof of Lemma 2 is easily obtained by the theorem on the logarithmic derivatives (see, [5] p. 245).

Lemma 3. *The differential equation* (1.1) *possesses no admissible solution for* $p=2$ *and* $q=1$.

For the proof of Lemma 3, we give a remark.

Remark 1. Let $\eta(z)$ be a rational of members of $\mathcal{M}$ and their derivatives. Then we have $T(r, \eta) \leq K \sum_{a_\nu \in \mathcal{M}} T(r, a_\nu) + S(r, w)$, for some $K > 0$. Thus $\eta(z)$ is a small function with respect to $w(z)$. We denote $n_\eta^*(r, c; w)$, the number of c-point z_0 of $w(z)$ in $|z| \leqq r$ so that z_0 satisfies $\eta(z_0)=0$. $N_\eta^*(r, c; w)$ is defined in the usual way. Assume that $N(r, c; w) \neq S(r, w)$, for some $c \in \boldsymbol{C} \cup \{\infty\}$, then there exists an admissible c-point of $w(z)$. Since $\eta(z)$ is small with respect to $w(z)$, there exists an admissible c-point of $w(z)$, which is neither zero nor pole of $\eta(z)$. Hence, if $N_\eta^*(r, c; w) \neq S(r, w)$, then $\eta(z) \equiv 0$.

Proof of Lemma 3. Suppose (1.1) possesses an admissible solution $w(z)$ for $p=2$ and $q=1$. Put $u = w + a_0(z)/a_1(z) + b_1(z)^2/4a_1(z)$, then

(2.2) $$(u' + \beta(z))^2 = a_1(z)u,$$

where $\beta(z) = b_1(z)/2 - (a_0(z)/a_1(z) + b_1(z)^2/4a_1(z))'$.

Suppose $N(r, 0; u) \neq S(r, u)$. Let z_0 be an admissible zero of $u(z)$. From (2.2), z_0 is a multiple zero of $u(z)$. Hence $u'(z_0)=0$, which implies

$\beta(z_0)=0$ by (2.2). Thus $N_\beta^*(r, 0; u)\neq S(r, u)$. By Remark 1, $\beta(z)\equiv 0$ and by Theorem A, (2.2) has no admissible solution. Therefore $N(r, 0; u)=S(r, u)$.

Put $\varphi(z)=u'/u$, then by Lemma 1

$$N(r, \varphi(z))\leqq N(r, u)+N(r, 0, u)=S(r, u).$$

By the theorem on the logarithmic derivatives, we have $m(r, \varphi)=S(r, u)$. Thus $\varphi(z)$ is a small function, hence $(\varphi(z)u+\beta(z))^2=a_1(z)u$, which implies $T(r, u)=S(r, u)$. This is a contradiction.

3. Proof of Theorem 1. For the proof of Theorem 1, we will follow Steinmetz's ideas in [7].

Proof of Theorem 1. By Lemma 3, we will prove for the case $p\geqq 3$. Suppose (1.1) possesses an admissible solution $w(z)$.

We consider the following conditions, for a complex constant τ.

$$m(r, \tau; w)=S(r, w), \tag{3.1}$$

and

$$Q(z, \tau)\not\equiv 0. \tag{3.2}$$

We have a plenty of such τ's as seen by the second fundamental theorem.

Put

$$F(z; \tau_j)=(P(z, w')-Q(z, \tau_j))/(w-\tau_j),$$

where τ_j $(j=1, 2, \cdots, p)$ are arbitrarily given distinct complex constants satisfying the conditions (3.1) and (3.2). Then by Lemma 1

$$N(r, F(z; \tau_j))=S(r, w) \quad j=1, 2, \cdots, p. \tag{3.3}$$

We consider a linear combinations $h(z)=\sum_{j=1}^p A_jF(z; \tau_j)$, A_j constants:

$$h(z)=P(z, w')\sum_{j=1}^p \frac{A_j}{w-\tau_j}-\sum_{j=1}^p \frac{A_jQ(z, \tau_j)}{w-\tau_j}. \tag{3.4}$$

From (3.3) we have $N(r, h)=S(s, w)$. By the condition (3.1), the proximity function of the second term of the right-hand side of (3.4) is $S(r, w)$. We choose complex constants $A_1, \cdots, A_p$ so that

$$\sum_{j=1}^p \frac{A_j}{w-\tau_j}=\frac{A}{\prod_{j=1}^p(w-\tau_j)}, \tag{3.5}$$

where A is a non-zero constant. In fact, this choice is regarded as a non-trivial solution of the system

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1^{(1)} & \sigma_2^{(1)} & \cdots & \sigma_p^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1^{(p-2)} & \sigma_2^{(p-2)} & \cdots & \sigma_p^{(p-2)} \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ \vdots \\ A_p \end{bmatrix}=0,$$

where $\sigma_n^{(m)}$ is a fundamental symmetric expression of τ_j $(1\leqq j\leqq p,\ j\neq n)$ of degree m. By (3.5) and Lemma 2, the proximity function of the first term of (3 4) is also $S(r, w)$. Thus we obtain $m(r, h)=S(r, w)$. Hence $h(z)$ is a small function with respect to $w(z)$.

First we treat the case $h(z)\not\equiv 0$. From (3.5)

$$AP(z, w')=h(z)\prod_{j=1}^{p}(w-\tau_j)+\sum_{j=1}^{p}A_jQ(z, \tau_j)\prod_{\nu\neq j}(w-\tau_\nu). \tag{3.6}$$

From (3.6), $T(r, w')=T(r, w)+S(r, w)$. Thus, by (1.3), $T(r, w)=S(r, w)$, which is a contradiction.

It remains to consider the case $h(z)\equiv 0$. From (3.5)

$$P(z, w')=\frac{1}{A}\sum_{j=1}^{p}A_jQ(z, \tau_j)\prod_{\nu\neq j}(w-\tau_\nu). \tag{3.7}$$

From (3.7) and (1.1), the right-hand side of (3.7) is identically equal to $Q(z, w)$, otherwise we have $T(r, w)=S(r, w)$. Comparing the coefficients of w^q, we obtain the following equation

$$t_1(z)A_1+t_2(z)A_2+\cdots+t_p(z)A_p=0, \tag{3.8}$$

where $t_j(z)=Q(z, \tau_j)-(-1)^{p-1}a_{p-1}(z)\prod_{\nu\neq j}\tau_\nu$ and $a_m(z)=0$, if $m\geqq q$. Since $p\geqq 3$ and $a_q(z)\not\equiv 0$, we choose τ_j $(j=1, 2, \cdots, p)$ so that

$$\det\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \sigma_1^{(1)} & \sigma_2^{(1)} & \cdots & \sigma_p^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_1^{(p-2)} & \sigma_2^{(p-2)} & \cdots & \sigma_p^{(p-2)} \\ t_1(z) & t_2(z) & \cdots & t_p(z) \end{bmatrix}\not\equiv 0.$$

Thus $A_j=0$ for all $j=1, 2, \cdots, p$, which contradicts our assumption. Hence Theorem 1 is proved.

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