69. *A New One-parameter Family of 2×2 Quantum Matrices*

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We introduce a new one-parameter family of quadratic braided 2×2 matrix bialgebras $B_q(2)$. We work over the complex numbers C. All proofs of this announcement will be included in [5]. The main results were also announced at the AMS San Fransisco meeting in January 1991.

We start with the following R-matrix. Let q be a complex number.

$$R_q=\left[1-\frac{(q-1)^2}{2}\right]e_{11}\otimes e_{11}+\left[1-\frac{(q+1)^2}{2}\right]e_{22}\otimes e_{22}$$
$$+\frac{(q-1)^2}{2}e_{12}\otimes e_{12}+\frac{(q+1)^2}{2}e_{21}\otimes e_{21}$$
$$+\frac{1-q^2}{2}(e_{11}\otimes e_{22}+e_{22}\otimes e_{11})+\frac{1+q^2}{2}(e_{12}\otimes e_{21}+e_{21}\otimes e_{12})$$

where e_{ij} denote the matrix units. A tedious verification shows that R_q satisfies the Yang-Baxter equation (or the braid condition)

$$(I\otimes R_q)(R_q\otimes I)(I\otimes R_q)=(R_q\otimes I)(I\otimes R_q)(R_q\otimes I).$$

Further we have $(R_q-I)(R_q+q^2I)=0$ and when $q\neq 0$, $q^2\neq -1$, R_q is diagonal with two two-dimensional eigenspaces.

Definition 1. Assume $q\neq 0$, $q^2\neq -1$. Let $B_q(2)$ be the C-algebra defined by generators a, b, c, d and the following relations

(1) $ad=da$, (2) $bc=cb$, (3) $ab-\hat{q}ba=(1-\hat{q})cd$,

(4) $dc+\hat{q}cd=(1+\hat{q})ba$, (5) $ac-\hat{q}ca=-(1+\hat{q})bd$,

(6) $db+\hat{q}bd=-(1-\hat{q})ca$, (7) $a^2+b^2=c^2+d^2$,

(8) $(1+\hat{q})b^2=(\hat{q}-1)c^2$,

where $\hat{q}=\dfrac{q+q^{-1}}{2}$.

The above relations are equivalent to saying that the matrix $X\otimes X$ with $X=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$, commutes with R_q. Hence the algebra $B_q(2)$ has a bialgebra structure with comultiplication

$$\Delta\begin{pmatrix} a & b \\ c & d\end{pmatrix}=\begin{pmatrix} a\otimes 1 & b\otimes 1 \\ c\otimes 1 & d\otimes 1\end{pmatrix}\begin{pmatrix} 1\otimes a & 1\otimes b \\ 1\otimes c & 1\otimes d\end{pmatrix}.$$

The bialgebra $B_q(2)$ is braided by [2] or [1].

Proposition 2. *Assume $q\neq 0$, $q^4\neq 1$. Let*

$$f=\frac{1}{2}(a+d),\quad g=\frac{1}{2}(a-d),\quad s=\frac{1}{2}(q_-b+q_+c),\quad t=\frac{1}{2}(q_-b-q_+c)$$

where $q_\pm=(\sqrt{q}\pm\sqrt{q}^{-1})^{-1}$.

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(a) *The algebra $B_q(2)$ is presented by generators f, g, s, t and the relations*

$$fg=gf=s^2+t^2,\quad st=ts=0,\quad fs=qsg,\quad sf=qgs,\quad tg=qft,\quad gt=qtf.$$

(b) *We have*

$$\begin{aligned}
&\Delta(f)=f\otimes f+g\otimes g+(q-q^{-1})(s\otimes s-t\otimes t),\\
&\Delta(g)=f\otimes g+g\otimes f+(q-q^{-1})(t\otimes s-s\otimes t),\\
&\Delta(s)=f\otimes s+g\otimes t+s\otimes f-t\otimes g,\\
&\Delta(t)=f\otimes t+g\otimes s+t\otimes f-s\otimes g,\\
&\varepsilon(f)=1,\quad \varepsilon(g)=\varepsilon(s)=\varepsilon(t)=0.
\end{aligned}$$

We are interested in representations and co-representations of $B_q(2)$. We assume $q\neq 0$ and $q^4\neq 1$ throughout.

Proposition and Definition 3. *For complex numbers ξ, η there is a representation $B_q(2)\to M_2(\boldsymbol{C})$ such that*

$$f\mapsto\frac{1}{2}\begin{pmatrix}\xi+\sqrt{\eta} & \\ & \xi-\sqrt{\eta}\end{pmatrix},\quad g\mapsto\frac{q}{2}\begin{pmatrix}\xi-\sqrt{\eta} & \\ & \xi+\sqrt{\eta}\end{pmatrix},$$

$$s\mapsto 0,\quad t\mapsto\frac{\sqrt{q}}{2}\begin{pmatrix} & \sqrt{\xi^2-\eta}\\ \sqrt{\xi^2-\eta} & \end{pmatrix}.$$

Let this representation be $\pi(\xi,\eta)$. Let $\pi'(\xi,\eta)=\pi(\xi,\eta)\circ\iota$ with ι the automorphism of $B_q(2)$, $\iota(f)=g$, $\iota(g)=f$, $\iota(s)=t$, $\iota(t)=s$.

Theorem 4 (cf. [6, Thm. 1]). (a) *All irreducible representations of $B_q(2)$ have dimension ≤ 2.*

(b) *$\pi(\xi,\eta)$ and $\pi'(\xi,\eta)$ with $\xi^2\neq\eta$, $\eta\neq 0$, give a complete set of representatives for all 2-dimensional irreducible representations of $B_q(2)$.*

Let $B_q(2)^\circ$ be the dual bialgebra of $B_q(2)$ [4].

Corollary 5. *The coradical of $B_q(2)^\circ$ is the direct sum of copies of $\boldsymbol{C}$ and $M_2(\boldsymbol{C})^*$.*

Definition 6. Let $F_q(\xi,\eta)$ be the subalgebra of $B_q(2)^\circ$ generated by the coefficient space for $\pi(\xi,\eta)$.

Main Theorem 7. *Let q, q', ξ be non-zero complex numbers. Assume neither q nor q' is a root of 1.*

(a) *The bialgebra $F_q(\xi,\xi^2q'^2)$ is cosemisimple, i.e., it is contained in the coradical of $B_q(2)^\circ$.*

(b) *The bialgebra map $B_q(2)\to F_q(\xi,\xi^2q'^2)^\circ$ corresponding to the inclusion $F_q(\xi,\xi^2q'^2)\to B_q(2)^\circ$ is injective.*

(c) *There is a bialgebra isomorphism*

$$B_{q'}(2)\simeq F_q(\xi,\xi^2q'^2)$$

such that the composite

$$B_q(2)\to F_q(\xi,\xi^2q'^2)^\circ\simeq B_{q'}(2)^\circ$$

has image $F_{q'}(\xi,\xi^2q^2)$.

In general, for coalgebras C and C' there is 1–1 correspondence among

(1) a linear map $\phi: C\to C'^*$,

(2) a linear map $\phi': C'\to C^*$,

(3) a bialgebra map $\psi: T(C)\to T(C')^\circ$,

(4) a bialgebra map $\psi': T(C')\to T(C)^\circ$,

(5) a bialgebra pairing $\chi: T(C)\times T(C')\to \boldsymbol{C}$.

Let $C=C'=M_2(\boldsymbol{C})^*$ with canonical base x_{ij}, $1\le i, j\le 2$, and let q, q', ξ as before. Take as ϕ of (1) the following map $\phi_\xi(q, q')$:

$$x_{11}\mapsto \frac{\xi}{2}\begin{pmatrix}1+q+q'-qq' & \\ & 1+q-q'+qq'\end{pmatrix},$$
$$x_{12}\mapsto \frac{\xi}{2}\begin{pmatrix} & -(1-q)(1-q') \\ -(1-q)(1+q') & \end{pmatrix},$$
$$x_{21}\mapsto \frac{\xi}{2}\begin{pmatrix} & -(1+q)(1-q') \\ -(1+q)(1+q') & \end{pmatrix},$$
$$x_{22}\mapsto \frac{\xi}{2}\begin{pmatrix}1-q+q'+qq' & \\ & 1-q-q'-qq'\end{pmatrix}.$$

Then we have

(i) $\phi'=\phi_\xi(q', q)$,

(ii) ψ *factors through* $B_q(2)$ *via* $\begin{pmatrix}x_{11} & x_{12}\\ x_{21} & x_{22}\end{pmatrix}\mapsto\begin{pmatrix}a & b\\ c & d\end{pmatrix}$.

(iii) ψ' *has image* $F_q(\xi, \xi^2 q'^2)$.

It follows that ψ and ψ' induce bialgebra maps

$$B_q(2)\to F_{q'}(\xi, \xi^2 q^2) \quad\text{and}\quad B_{q'}(2)\to F_q(\xi, \xi^2 q'^2).$$

Theorem 7 (c) means these are isomorphisms.

Corollary 8. (a) *If* q $(\neq 0)$ *is not a root of* 1, $B_q(2)$ *is co-semisimple. It is the direct sum of* $\boldsymbol{C}$ *and copies of* $M_2(\boldsymbol{C})^*$.

(b) *If* $q, q'\in k-\{0\}$ *are not roots of* 1, *there is a non-degenerate bialgebra pairing*

$$B_q(2)\times B_{q'}(2)\to \boldsymbol{C}.$$

Let q $(\neq 0)$ be not a root of 1, and let $\hat{q}=\frac{1}{2}(q+q^{-1})$. Let $S=\boldsymbol{C}[x, y]/((1-\hat{q})x^2-(1+\hat{q})y^2)$ which is isomorphic to $\boldsymbol{C}[x, y]/(xy)$ by a linear change of generators.

Lemma 9. *The map* $x\to x\otimes a+y\otimes c$ *and* $y\mapsto x\otimes b+y\otimes d$ *makes* S *into a right* $B_q(2)$*-comodule algebra.*

Let $S^!$ be the Manin dual of S [3]. It is a left $B_q(2)$-comodule algebra. The Koszul complex (ibid.)

$$\cdots\to S\otimes S_n^{!*}\xrightarrow{\partial} S\otimes S_{n-1}^{!*}\to\cdots\to S\otimes S_0^{!*}\to \boldsymbol{C}\to 0$$

consists of right $B_q(2)$-comodules and comodule maps.

Theorem 10. *The Koszul complex is exact. The trivial comodule* $\boldsymbol{C}$ *and* $\mathrm{Im}(\partial: S_m\otimes S_{n+1}^{!*}\to S_{m+1}\otimes S_n^{!*})$, $m, n\ge 0$, *form a complete set of simple* $B_q(2)$*-comodules.*

Here, $(\ )_n$ denotes the degree n part.

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