61. A Note on Poincaré Sums of Galois Representations. II

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Let k be a field of characteristic zero, K a finite Galois extension of k and χ the character of a k-representation ρ of the Galois group $G=G(K/k)$. The subfield corresponding to Ker ρ is written K_χ because Ker $\rho=$ Ker $\chi^* \overset{\mathrm{def}}{=} \{s \in G\,;\, \chi^*(s)=1\}$ where we set $\chi^*(s)=\chi(s)/\chi(1)$. In [5], we proved

$$K_\chi = k(P_\chi) \tag{0.1}$$

where

$$P_\chi = \sum_{s\in G} \theta^s \chi(s) \qquad \text{(a Poincaré sum)} \tag{0.2}$$

and θ is a normal basis element for K/k, chosen once for all.

If, in particular, K/k is a cyclic Kummer extension of degree n with $G=\langle s\rangle$, $\rho(s)=\zeta$, this being a primitive nth root of 1 in k, then $K=k(P_\chi)$ as well as $P_\chi^n \in k$, a property peculiar to this K/k. Usually P_χ is referred to as the Lagrange resolvent and satisfies

$$P_\chi^s = \chi(s^{-1})P_\chi. \tag{0.3}$$

Therefore, it is natural to seek a generalization of (0.3) for any Galois extension K/k such that G splits over k[1].

In this paper, we shall prove among others that

$$P_\chi^{a(s)} = \chi^*(s^{-1})P_\chi, \quad s \in G, \quad \chi \in \mathrm{Irr}\,(G)^{2)} \tag{0.4}$$

where

$$a(s) = \frac{1}{n}\sum_{t\in G} tst^{-1}, \qquad n=[K:k]=|G|. \tag{0.5}$$

This $a(s)$ is an element of the center $k[G]_0$ of the group ring $k[G]$ and is viewed as an endomorphism of the vector space K over k. When k is a number field, (0.4) implies that

$$P_\chi^{\alpha_{K/k}(\mathfrak{p})} = \chi^*\left(\left[\frac{K/k}{\mathfrak{P}}\right]^{-1}\right)P_\chi, \qquad \mathfrak{P}\,|\,\mathfrak{p}, \tag{0.6}$$

where $\alpha_{K/k}$ is the generalized Artin map introduced and studied in the series of papers [2], [3], [4].

1. Operator $a(s)$. Let K/k be a finite Galois extension of fields of characteristic zero with $G=G(K/k)$. Fix once for all a normal basis element θ for K/k. Assume that k is a splitting field for G. We begin with a description of the following diagram

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1) By a theorem of Brauer ([1], p. 86, (16.3)), this is always the case if k contains a primitipe mth root of 1 where m is the exponent of G.

2) Irr (G) denotes the set of all absolutely irreducible characters of G.

$$
(1.1)\qquad
\begin{array}{ccccc}
K & \xrightarrow[\sim]{\varphi} & k[G] & \xrightarrow[\sim]{\rho} & \bigoplus_{\nu=1}^{r} M_{n_\nu}(k) \\
 & & \uparrow i & & \uparrow j \\
 & & k[G]_0 & \xrightarrow[\sim]{\omega} & k^r \\
 & & \uparrow & & \\
 & & M(G)=\langle\gamma_1,\cdots,\gamma_r\rangle. & &
\end{array}
$$

The map φ is an isomorphism of $k[G]$-modules given by

$$(1.2)\qquad \varphi(\sum_{s\in G} a_s\theta^s)=\sum_{s\in G} a_s s, \qquad a_s\in k.$$

The map ρ is an isomorphism of k-algebras given by $\rho=(\rho_\nu)$ where ρ_ν, $1\leqq\nu\leqq r$, are the distinct irreducible representations of $k[G]$ with $n_\nu=\deg\rho_\nu=\chi_\nu(1)$. The maps i, j are natural embeddings of centers of k-algebras $k[G]$, $\bigoplus_{\nu=1}^r M_{n_\nu}(k)$, respectively. The map ω is an isomorphism of commutative k-algebras given by

$$(1.3)\qquad \omega=(\omega_\nu),\quad \omega_\nu(z)=\frac{1}{n_\nu}\chi_\nu(z),\quad \chi_\nu\in\mathrm{Irr}\,(G).$$

One verifies that $\rho i=j\omega$. If $\{s_i\}$, $1\leqq i\leqq r$, is a complete set of representatives of conjugacy classes of G, we put $\gamma_i=a(s_i)$ and denote by $M(G)$ the multiplicative monoid generated by γ_i's in $k[G]_0$. Via φ, we view K as a $k[G]_0$-module. For example, we have

$$(1.4)\qquad x^{a(s)}=\frac{1}{n}\sum_{t\in G} x^{tst^{-1}}, \qquad x\in K,\quad s\in G.$$

From (0.5), (1.3), it follows that

$$(1.5)\qquad \omega(a(s))=(\chi_\nu^*(s)), \qquad 1\leqq\nu\leqq r.$$

For each $\chi\in\mathrm{Irr}\,(G)$, we set

$$(1.6)\qquad E_\chi=\{x\in K\,;\, x^{a(s)}=\chi^*(s^{-1})x \text{ for all } s\in G\}.$$

(1.7) **Theorem.** *E_χ, $\chi\in\mathrm{Irr}\,(G)$, are $k[G]_0$-submodules of K and we have $K=\bigoplus_\chi E_\chi$, $\dim E_\chi=\chi(1)^2$.*

Proof. For each $\chi\in\mathrm{Irr}\,(G)$, set

$$(1.8)\qquad S_\chi=\rho\varphi(E_\chi).$$

In view of (1.5), (1.6), we have

$$(1.9)\qquad S_\chi=\Big\{X=(X_\nu)\in\bigoplus_{\nu=1}^{r} M_{n_\nu}(k)\,;\,\chi_\nu^*(s)X_\nu=\chi^*(s^{-1})X_\nu,\ \forall s\in G\Big\}.$$

For $\chi\in\mathrm{Irr}\,(G)$, denote by χ^c the irreducible character given by $\chi^c(s)=\chi(s^{-1})$.[3] For μ, $1\leqq\mu\leqq r$, define μ^c by the equality $\chi_{\mu^c}=(\chi_\mu)^c$. Call p_λ, $1\leqq\lambda\leqq r$, the projection $\bigoplus_{\nu=1}^r M_{n_\nu}(k)\to M_{n_\lambda}(k)$. Then, (1.9) becomes

$$(1.10)\qquad S_{\chi_\mu}=\{X=(X_\nu)\,;\,X_\nu^*(s)X_\nu=\chi_{\mu^c}^*(s)X_\nu \text{ for all } s\in G\}$$

and we get

$$p_\nu(S_{\chi_\mu})=\begin{cases} M_{n_\mu}(k) & \text{if } \nu=\mu^c,\\ 0 & \text{if } \nu\neq\mu^c,^{4)}\end{cases}$$

from which our assertion follows, Q.E.D.

(1.11) **Corollary.** *The characteristic polynomial of* $a(s)\in\mathrm{End}_k(K)$ *is*

[3] If $\chi=\mathrm{tr}\,\rho$, then $\chi^c=\mathrm{tr}\,\rho^c$ with $\rho^c(s)={}^t\rho(s)^{-1}$.

[4] Note that the set $\{\chi_\nu\}$ is independent over k.

given by

$$X(a(s);T)=\prod_{\chi\in\mathrm{Irr}(G)}(T-\chi^*(s^{-1}))^{\chi(1)^2}.$$

(1.12) **Theorem.** *For each $\chi\in\mathrm{Irr}(G)$, we have $P_\chi\in E_\chi$, i.e., the Poincaré sum P_χ is a simultaneous eigen-vector for all operators in the commutative monoid $M(G)$.*

Proof. Let ρ_ν be an irreducible k-representation of G such that $\mathrm{tr}\,\rho_\nu=\chi_\nu$ and let $\rho_\nu(t)=(\rho_{\nu,ij}(t))\in M_{n_\nu}(k)$. By the orthogonality relation, we have

(1.13) $$\sum_{t\in G}\rho_{\mu,ij}(t)\rho_{\nu,kl}(t)=0 \qquad \text{if } \mu^c\neq\nu,$$

or

(1.14) $$\sum_{t\in G}\rho_{\mu,ij}(t)\rho_\nu(t)=0,\quad 1\leqq i,\, j\leqq n_\mu, \qquad \text{if } \mu^c\neq\nu.$$

Summing up (1.14) for (i, j) with $i=j$, we get

(1.15) $$\sum_{t\in G}\chi_\mu(t)\rho_\nu(t)=0 \qquad \text{if } \mu^c\neq\nu.$$

Therefore, we have $Q_{\chi_\mu}=\rho\varphi(P_{\chi_\mu})=\rho\varphi(\sum_{s\in G}\chi_\mu(s)\theta^s)=\rho(\sum_{s\in G}\chi_\mu(s)s)=(X_\nu)\in\bigoplus_{\nu=1}^r M_{n_\nu}(k)$ with

(1.16) $$X_\nu=\sum_{t\in G}\chi_\mu(t)\rho_\nu(t)=0 \qquad \text{if } \mu^c\neq\nu.$$

On the other hand, if $\mu^c=\nu$, then we have

(1.17) $$\chi_\nu^*(s)X_\nu=\chi_\mu^*(s^{-1})X_\nu.$$

From (1.16), (1.17), it follows that $Q_{\chi_\mu}\in S_{\chi_\mu}$ and hence $P_{\chi_\mu}\in E_{\chi_\mu}$, Q.E.D.

2. $\alpha_{K/k}$. Let K/k be a finite Galois extension of number fields, S a finite set of fiinite primes of k containing all primes which ramify in K and $I^+(S)$ the free commutative monoid generated by primes $\mathfrak{p}\notin S$. The map $\alpha_{K/k}: I^+(S)\to M(G)$ is a surjective monoid homomorphism given by

(2.1) $$\alpha_{K/k}(\mathfrak{p})=a(F_\mathfrak{p}), \qquad \mathfrak{P}\,|\,\mathfrak{p},$$

where $F_\mathfrak{p}=\left[\dfrac{K/k}{\mathfrak{P}}\right]$, the Frobenius automorphism of $\mathfrak{P}$. If we put, for $\chi\in\mathrm{Irr}\,(G)$,

(2.2) $$(\chi,\mathfrak{a})=\prod_{\mathfrak{p}}\chi^*(F_\mathfrak{p}^{-1})^{e_\mathfrak{p}} \qquad \text{for } \mathfrak{a}=\prod_{\mathfrak{p}}\mathfrak{p}^{e_\mathfrak{p}}\in I^+(S),$$

then, by (1.12), (2.1), we have

(2.3) $$P_\chi^{\alpha_{K/k}(\mathfrak{a})}=(\chi,\mathfrak{a})P_\chi,\quad \mathfrak{a}\in I^+(S),\quad \chi\in\mathrm{Irr}\,(G).$$

In other words, the Poincaré sum is a simultaneous eigen-vector for all operators $\alpha_{K/k}(\mathfrak{a})$, $\mathfrak{a}\in I^+(S)$.

(2.4) **Remark.** We hope to come back to the study of eigen-values (2.2) sometime in the future.

References

[1] Feit, W.: Characters of Finite Groups. Benjamin, New York, Amsterdam (1967).
[2] Ono, T.: A note on the Artin map. Proc. Japan Acad., **65A**, 304–306 (1989).
[3] ——: A note on the Artin map. II. ibid., **66A**, 132–136 (1990).
[4] ——: A note on the Artin map. III. ibid., **67A**, 79–81 (1991).
[5] ——: A note on Poincaré sums of Galois representations. ibid., **67A**, 145–147 (1991).