46. *The Flat Holomorphic Conformal Structure on the Horrocks-Mumford Orbifold*

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We construct the explicit flat holomorphic conformal structure on an orbifold. We often abbreviate 'Horrocks-Mumford' to 'HM', and 'holomorphic conformal structure' to 'HCS'. $P_n(C)$ denotes the n-dimensional complex projective space.

In the paper [2], Horrocks and Mumford constructed a holomorphic vector bundle $\mathcal{F}_{HM}$ of rank two on the $P_4(C)$. The space $\Gamma\mathcal{F}_{HM}$ of its sections is four-dimensional. If the zero set X_s of a section $s \in \Gamma\mathcal{F}_{HM}$ is a smooth surface, X_s is an abelian surface with (1.5)-polarization. In fact, they proved that $P_3(C)=P(\Gamma\mathcal{F}_{HM})$ is birationally equivalent to the moduli space $\mathcal{A}_{1,5}$ of the abelian surfaces with (1, 5)-polarization and level-5-structure. (See [2] [4].) We call this projective space *the HM-orbifold.*

While the moduli space $\mathcal{A}_{1,5}$ is realized as a quotient space $\mathcal{H}_2/\Gamma_{1,5}$ of the Siegel upper space $\mathcal{H}_2$ of degree two. Here $\Gamma_{1,5}$ is a certain discrete subgroup of $Sp(4, \boldsymbol{R})$. (See [4].) $\mathcal{H}_2$ is embedded in a non-degenerate hyperquadrics $\{[z_0 : z_1 : z_2 : z_3 : z_4] \in P_4(C)\,;\ \sum_{0\leqq i,j\leqq 4} a_{ij}z_iz_j=0\}$. The holomorphic tensor field $\phi=\sum_{0\leqq i,j\leqq 4} a_{ij}dz_idz_j$ on $\mathcal{H}_2$ is conformally flat and its conformally class is invariant under the automorphisms of $\mathcal{H}_2$. Therefore ϕ induces a tensor φ on the HM-orbifold which is called *the flat HCS.* Applying a higher dimensional version of Kobayashi and Naruki's theory in [3], we can calculate the flat HCS.

Theorem 1. *Let p be the projection $C^4\backslash\{0\}\to P_3(C)$. The pullback of the flat HCS φ is given by in homogeneous coordinates*

$$p^*\varphi=\sum_{0\leqq i,j\leqq 3} g_{ij}dx_idx_j$$

where

$$g_{00}=2(-x_0^2x_1x_2-x_0^2x_3^2+x_0x_1^3+2x_0x_2^2x_3+2x_1^2x_2x_3-3x_1x_3^3)$$
$$g_{01}=x_0^3x_2-2x_0^2x_1^2-7x_0x_1x_2x_3+4x_0x_3^3+x_1^3x_3+4x_1x_2^3-5x_2^2x_3^2$$
$$g_{02}=x_0^3x_1-x_0^2x_2x_3-x_0x_1^2x_3-4x_1^2x_2^2+5x_1x_2x_3^2$$
$$g_{03}=2x_0^3x_3-3x_0^2x_2^2+4x_0x_1^2x_2+2x_0x_1x_3^2-x_1^4$$
$$g_{11}=2(x_0^3x_1+2x_0^2x_2x_3-x_0x_1^2x_3-3x_0x_2^3-x_1^2x_2^2+2x_1x_2x_3^2)$$
$$g_{12}=-x_0^4+4x_0^2x_1x_3+2x_0x_1x_2^2+2x_1^3x_2-3x_1^2x_3^2$$
$$g_{13}=-x_0^2x_1x_2-4x_0^2x_3^2+x_0x_1^3+5x_0x_2^2x_3-x_1^2x_2x_3$$
$$g_{22}=2(-x_0^3x_3+x_0x_1^2x_2+5x_0x_1x_3^2-x_1^4)$$
$$g_{23}=3(x_0^3x_2-x_0^2x_1^2-5x_0x_1x_2x_3+x_1^3x_3)$$
$$g_{33}=2(-x_0^4+x_0^2x_1x_3+5x_0x_1x_2^2-x_1^3x_2)$$
$$g_{10}=g_{01},\ g_{20}=g_{02},\ g_{30}=g_{03},\ g_{21}=g_{12},\ g_{31}=g_{13},\ g_{32}=g_{23}.$$

This flat HCS is degenerate along the trisecant surface to the rational sextic curve $C: (5\lambda^4 : 5\lambda^2 : \lambda^6 - 2\lambda : 2\lambda^5 + 1)$. (See [1].)

Remark. A Hilbert modular surface for $\boldsymbol{Q}(\sqrt{5})$ is embedded in the HM-orbifold as the diagonal cubic of Clebsch. Pullback of the flat HCS to the cubic surface gives the HCS obtained in [3, (6.3)].

We quote a theorem from [5].

Theorem 2 ([5, Theorem 2.5]). *Assume the dimension of the space* $= n \geqq 3$. *Let* $\sum \sigma_{ij} dx^i dx^j$ *be a non-degenerate symmetric tensor which is conformally flat. Then the system*

$$\sigma_{ij}\left(w_{kl} - \sum_p \Gamma^p_{kl} w_p + \frac{1}{n-2} R_{kl} w\right) = \sigma_{kl}\left(w_{ij} - \sum_p \Gamma^p_{ij} w_p + \frac{1}{n-2} R_{ij} w\right)$$

is of rank $n+2$ *and ratio* $[s_0 : \cdots : s_{n+1}]$ *of its linearly independent solutions takes its values in a hyperquadrics. Here* Γ^i_{jk} *and* R_{ij} *stand for the Christoffel symbol and the Ricci tensor with respect to* σ_{ij} *and* w_i *is the derivative of* w *with respect to* x_i.

As a corollary, we obtain the explicit form of the uniformizing differential equation of the HM-orbifold in the sense of Yoshida [6]. However, we have to omit the Christoffel symbol and the Ricci tensor with respect to the flat HCS g_{ij} because they are far from simple.

References

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