5. A Necessary Condition for Monotone (P, μ)-u.d. mod 1 Sequences

By Kazuo GOTO*) and Takeshi KANO**)

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Abstract: Schatte [2: assertion (15)] remarked that
$$\lim_{n\to\infty} g(n)/\log n=\infty,$$
if the sequence $(g(n))$ is non-decreasing and uniformly distributed in the ordinary sense. Niederreiter proved ([1] Theorem 2) that:

Let μ be a Borel probability measure on R/Z that is not a point measure and let p be a weighted means. If $(g(n))$ is a non-decreasing (P, μ)-u.d. mod 1 sequence, then necessarily

$$(*) \qquad \lim_{n\to\infty} g(n)/\log s(n)=\infty,$$

where $s(n)=p(1)+p(2)+\cdots+p(n)$ is such that $s(n)\uparrow\infty$.

In this paper we shall prove $(*)$ along the same lines as Schatte.

§ 1. **Definitions.** Let $P=(p(n))$, $n=1, 2, \cdots$, be a sequence of non-negative real numbers with $p(1)>0$. For $N\geqq 1$, we put $s(N)=p(1)+p(2)+\cdots+p(N)$ and assume throughout that $s(N)\to\infty$ as $N\to\infty$.

We define after Tsuji [3] the $(M, p(n))$-u.d. mod 1.

Definition 1. A sequence $(g(n))$ is said to be $(M, p(n))$-uniformly distributed mod 1 (or shortly $(M, p(n))$-u.d. mod 1), if

$$(1) \qquad \lim_{N\to\infty} \frac{1}{s(N)} \sum_{n=1}^{N} p(n)C_J(\{g(n)\})=|J|,$$

holds for all intervals J in $\boldsymbol{R}/\boldsymbol{Z}$. Here C_J denotes the characteristic function of J.

It is known that an alternative definition is as follows:

A sequence $(g(n))$ is said to be $(M, p(n))$-u.d. mod 1 if for all positive integers h,

$$\lim_{N\to\infty} \frac{1}{s(N)} \sum_{n=1}^{N} p(n)e^{2\pi i h g(n)}=0.$$

We define after Niederreiter [1] the (P, μ)-u.d. mod 1 as follows:

Definition 2. Let $(p(n))$ and $(s(n))$ be sequences of Definition 1 and μ be a Borel probability measure on $\boldsymbol{R}/\boldsymbol{Z}$. Then a sequence $(g(n))$ is said to be (P, μ)-uniformly distributed mod 1 (or shortly (P, μ)-u.d. mod 1), if

$$(2) \qquad \lim_{N\to\infty} \frac{1}{s(N)} \sum_{n=1}^{N} p(n)C_J(\{g(n)\})=\mu(J),$$

holds for all J in $\boldsymbol{R}/\boldsymbol{Z}$. Or equivalently, a sequence $(g(n))$ is said to be (P, μ)-u.d. mod 1 if

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*) Information Science, Sakuyo Junior College.

**) Department of Mathematics, Faculty of Science, Okayama University.

$$\lim_{N\to\infty}\frac{1}{s(N)}\sum_{n=1}^{N}p(n)e^{2\pi i h g(n)}=\int_0^1 e^{2\pi i h x}d\mu(x).$$

holds for all positive integers h.

§2. **Theorems.** **Theorem 1.** *Let $(g(n))$ be a non-decreasing real sequence.*

If $(g(n))$ is $(M, p(n))$-u.d. mod 1, then

$$\lim_{n\to\infty}\frac{g(n)}{\log s(n)}=\infty.$$

Proof. Since $g(n)$ is $(M, p(n))$-u.d. mod 1, for any given $\varepsilon>0$ there exists an N such that for all $n\geqq N$,

$$\frac{1}{s(n)}\left|\sum_{j=1}^{n}p(j)\exp(2\pi i g(j))\right|<\varepsilon.$$

For some fixed N, we can choose a non-decreasing positive real sequence $\{\nu(k)\}_{i=0}^{\infty}$, $\nu(0)=1$ such that $\nu(k)N$ are integers for all k,

$$s(\nu(k+1)N)\geqq s(\nu(k)N)A(\varepsilon)$$

and

$$s(\nu(k+1)N)<s(\nu(k)N)A(\varepsilon)^2,$$

where $A(\varepsilon)=(1/\sqrt{2}+\varepsilon)/(1/\sqrt{2}-\varepsilon)$.

Since for each $\nu(k)$,

$$\frac{1}{s(\nu(k)N)}\left|\sum_{j=1}^{\nu(k)N}p(j)\exp(2\pi i g(j))\right|<\varepsilon,$$

we have

$$\text{(3)}\quad \left|\sum_{j=\nu(k)N+1}^{\nu(k+1)N}p(j)\exp(2\pi i(g(j)-g(\nu(k)N)))\right|$$
$$=\left|\sum_{j=\nu(k)N+1}^{\nu(k+1)N}p(j)\exp(2\pi i(g(j)))\right|<\varepsilon(s(\nu(k+1)N)+s(\nu(k)N)).$$

To prove $g(\nu(k+1)N)-g(\nu(k)N))\geqq 1/8$ for all pairs (k, N), $k=0, 1, \cdots$; $N=1, 2, \cdots$, assume on the contrary, that there exists at least one pair (k, N) such that

$$0\leqq g(\nu(k+1)N)-g(\nu(k)N))<1/8.$$

If we consider the real part of (3), then we have

$$\left|\sum_{j=\nu(k)N+1}^{\nu(k+1)N}p(j)\cos(2\pi(g(j)-g(\nu(k)N)))\right|<\varepsilon(s(\nu(k+1)N)+s(\nu(k)N)).$$

Since $g(n)$ is non-decreasing, we have

$$0\leqq g(j)-g(\nu(k)N)\leqq g(\nu(k+1)N)-g(\nu(k)N).$$

Thus

$$\frac{1}{\sqrt{2}}(s(\nu(k+1)N)-s(\nu(k)N))<\varepsilon(s(\nu(k+1)N)+s(\nu(k)N)).$$

This contradicts to the definition of $(\nu(k))$.

Thus we obtain for $k=0, 1, 2, \cdots$, and every N,

$$\text{(4)}\quad g(\nu(k+1)N)-g(\nu(k)N))\geqq 1/8.$$

So we have by (4),

$$g(\nu(m)N)\geqq m/8+g(N).$$

On the other hand,

$$\log s(\nu(m)N) = \log \frac{s(\nu(m)N)}{s(\nu(m-1)N)} \cdot \frac{s(\nu(m-1)N)}{s(\nu(m-2)N)} \cdot \cdots \cdot \frac{s(\nu(1)N)}{s(\nu(0)N)} s(N)$$
$$\leqq \log A(\varepsilon)^{2m} s(N).$$

Thus for $\nu(m)N \leqq n < \nu(m+1)N$

$$\varliminf_{n\to\infty} \frac{g(n)}{\log s(n)} \geqq \varliminf_{m\to\infty} \frac{g(\nu(m)N)}{\log s(\nu(m+1)N)}$$
$$\geqq \varliminf_{m\to\infty} \frac{m/8 + g(N)}{(2m+2)\log A(\varepsilon) + \log s(N)} = \frac{1}{16 \log A(\varepsilon)}.$$

Since ε is arbitrarily small, we obtain

$$\varliminf_{n\to\infty} \frac{g(n)}{\log s(n)} = \infty,$$

which proves Theorem 1.

Theorem 2. *Let $(g(n))$ be a non-decreasing real sequence. If $g(n)$ is (P, μ)-u.d. mod 1, then*

$$\lim_{n\to\infty} \frac{g(n)}{\log s(n)} = \infty.$$

Proof. By the definition of a distribution function, if a random variable X has a distribution function $F(x)$, then $F(X)$ has a uniform distribution function, namely $x = \text{Prob}(X \leqq x)$. For $F(x) = \text{Prob}(X \leqq x)$ implies $F(x) = \text{Prob}(F(X) \leqq F(x))$ since $F(x)$ is increasing. Hence it follows $y = \text{Prob}(F(X) \leqq y)$ which means that $F(X)$ is uniformly distributed.

Now we define $F(x)$ with respect to Borel measure μ,

$$F(x) = \int_0^x d\mu = \mu([0, x)) \quad \text{on} \quad x \in [0, 1].$$

Also we define a sequence $G(n) = [g(n)] + F(\{g(n)\})$, where $[t]$ and $\{t\}$ denote the integral part of t and the fractional part of t, respectively. It follows that $G(n)$ is $(M, p(n))$-u.d. mod 1. From this fact, we have

$$\frac{G(n)}{\log s(n)} = \frac{[g(n)] + F(\{g(n)\})}{\log s(n)} \leqq \frac{g(n)+1}{\log s(n)}.$$

Since $s(n) \uparrow \infty$, we have by Theorem 1,

$$\varliminf_{n\to\infty} \frac{g(n)}{\log s(n)} \geqq \varliminf_{n\to\infty} \frac{G(n)}{\log s(n)} = \infty,$$

which proves Theorem 2.

References

[1] H. Niederreiter: Distribution mod 1 of monotone sequences, Indag. Math., **46**, 315–327 (1984).

[2] P. Schatte: On H_∞-summability and the uniform distribution of sequences. Math. Nachr., **113**, 237–243 (1983).

[3] M. Tsuji: On the uniform distribution of numbers mod 1. J. Math. Soc. Japan, **4**, 313–322 (1952).