43. Some Dolbeault Isomorphisms for Locally Trivial Fiber Spaces and Applications

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1. Let N be a paracompact complex manifold of complex dimension n, S a Stein manifold of complex dimension l and $\pi: M \to N$ a locally trivial holomorphic fiber space whose fibers are biholomorphic onto S. Put $m := \dim_C M (=n+l)$. Let $\{D_\alpha\}$ be a locally finite open covering of N satisfying that each D_α is a coordinate open subset with the trivialization $i_\alpha : \pi^{-1}(D_\alpha) \to D_\alpha \times S$ with $\prod_\alpha^1 \cdot i_\alpha = \pi$, where $\prod_\alpha^1$ denotes the projection $D_\alpha \times S \ni (a, b) \mapsto a \in D_\alpha$. Let $\{U_\sigma\}$ be a sufficiently fine and locally finite open covering of S so that each U_σ is biholomorphic onto a polydisc in C^l. We sometimes identify $\pi^{-1}(D_\alpha)$ with $D_\alpha \times S$. Let $z_\alpha = (z_\alpha^1, \cdots, z_\alpha^n)$ be a local coordinate defined on D_α and $w_\sigma = (w_\sigma^1, \cdots, w_\sigma^l)$ a local coordinate defined on U_σ. We put $\zeta_{\alpha,\sigma}^i = z_\alpha^i$ $(1 \le i \le n)$ and $\zeta_{\alpha,\sigma}^{n+j} = w_\sigma^j$ $(1 \le j \le l)$. Then $\zeta_{\alpha,\sigma} = (\zeta_{\alpha,\sigma}^1, \cdots, \zeta_{\alpha,\sigma}^{n+l}) = (z_\alpha^1, \cdots, z_\alpha^n, w_\sigma^1, \cdots, w_\sigma^l)$ defines a local coordinate in $i_\alpha^{-1}(D_\alpha \times U_\sigma)$. For an open subset $V \subset M$, we put $\mathcal{F}(V) := \{f \mid f$ is of class C^∞ in V and for any $z \in \pi(V)$, $f \mid \pi^{-1}(z) \cap V$ is holomorphic$\}$. We denote by $\mathcal{F}$ the sheaf defined by the presheaf $\{\mathcal{F}(V)\}$. Put $V_{\alpha,\sigma} := V \cap i_\alpha^{-1}(D_\alpha \times U_\sigma)$ and $\mathcal{F}^{r,p}(V) := \{\varphi \mid \varphi$ is a $C^\infty(r,p)$-form on V and $\varphi \mid V_{\alpha,\sigma} = \sum_{I,J} \varphi_{IJ} d\zeta_{\alpha,\sigma}^I \wedge d\bar{z}_\alpha^J$, $\varphi_{IJ} \in \mathcal{F}(V_{\alpha,\sigma})$ for each α and $\sigma\}$, where $I = (i_1, \cdots, i_r)$, $J = (j_1, \cdots, j_p)$, $1 \le i_1 < \cdots < i_r \le n+l$, and $1 \le j_1 < \cdots < j_p \le n$. We get the sheaf $\mathcal{F}^{r,p}$ defined by the presheaf $\{\mathcal{F}^{r,p}(V)\}$ for $0 \le r \le n+l$ and $0 \le p \le n$. Let Ω^r be the sheaf of germs of holomorphic r-forms on M. We have an exact sequence

$$0 \to \Omega^r \to \mathcal{F}^{r,0} \to \mathcal{F}^{r,1} \to \cdots \to \mathcal{F}^{r,n} \to 0.$$

For an open subset $W \subset \pi^{-1}(D_\alpha)$, put $\mathcal{E}_\alpha^{r,p,s}(W) := \{\psi \mid \psi$ is a C^∞ $(r, p+s)$-form in W and $\varphi \mid W \cap i_\alpha^{-1}(D_\alpha \times U_\sigma) = \sum_{I,J,K} \psi_{IJK} d\zeta_{\alpha,\sigma}^I \wedge d\bar{z}_\alpha^J \wedge d\bar{w}_\sigma^K$ for each $\sigma\}$, where $K = (k_1, \cdots, k_s)$ and $1 \le k_1 < \cdots < k_s \le l$. The presheaf $\{\mathcal{E}_\alpha^{r,p,s}(W)\}$ makes the sheaf $\mathcal{E}_\alpha^{r,p,s}$ on $\pi^{-1}(D_\alpha)$. Then we have an exact sequence

$$0 \to \mathcal{F}^{r,p} \mid \pi^{-1}(D_\alpha) \to \mathcal{E}_\alpha^{r,p,0} \to \mathcal{E}_\alpha^{r,p,1} \to \cdots \to \mathcal{E}_\alpha^{r,p,l} \to 0$$

for each α, where the mapping $\mathcal{E}_\alpha^{r,p,s} \to \mathcal{E}_\alpha^{r,p,s+1}$ is induced by the Cauchy-Riemann operator $\bar{\partial}_S$ on S. Solving the Cauchy-Riemann equation $\frac{\partial f(z,w)}{\partial \bar{w}_j} = g(z,w)$ with C^∞ parameter $z \in D_\alpha$ and using the standard argument for Dolbeault lemma, we can prove

$$H^q(D_\alpha \times U_\sigma, \mathcal{F}^{r,p}) \cong \frac{\{\varphi \in H^0(D_\alpha \times U_\sigma, \mathcal{E}_\alpha^{r,p,q}) \mid \bar{\partial}_S \varphi = 0\}}{\bar{\partial}_S H^0(D_\alpha \times U_\sigma, \mathcal{E}_\alpha^{r,p,q-1})} = 0$$

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for $q \geq 1$. This means the open covering $\{D_\alpha \times U_\sigma\}$ of $D_\alpha \times S$ is a Leray covering for the sheaf $\mathcal{F}^{r,p}$. We denote the Frechet space of all C^∞ functions on D_α by $E = C^\infty(D_\alpha)$. Then the cohomology group $H^q(\{D_\alpha \times U_\sigma\}, \mathcal{F}^{r,p})$ is isomorphic onto $H^q(\{U_\sigma\}, \Omega^r \varepsilon E)$ and $H^q(\pi^{-1}(D_\alpha), \mathcal{F}^{r,p}) = H^q(S, \Omega^r \varepsilon E) = 0$ by the result of Bungart [1] $q \geq 1$. This means the covering $\mathcal{W} = \{\pi^{-1}(D_\alpha)\}$ is a Leray covering for the sheaf $\mathcal{F}^{r,p}$ on M. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the covering $\{D_\alpha\}$ on N. For $\{f_{\alpha\beta}\} \in Z^1(\mathcal{W}, \mathcal{F}^{r,p})$ we put $g_\beta(p) := \sum_\alpha \rho_\alpha \cdot \pi(p) f_{\alpha\beta}(p)$ $p \in \pi^{-1}(D_\beta)$. Then $\{g_\alpha\} \in C^0(\mathcal{W}, \mathcal{F}^{r,p})$ and $\delta\{g_\alpha\} = \{f_{\alpha\beta}\}$. Hence $H^1(M, \mathcal{F}^{r,p}) = 0$. Similarly we have $H^q(M, \mathcal{F}^{r,p}) = 0$ $q \geq 1$.

Lemma. $H^q(M, \mathcal{F}^{r,p}) = 0$, $q \geq 1$.

By this lemma we obtain the following

Theorem 1.1.

$$H^p(M, \Omega^r) \cong \frac{\{\varphi \in H^0(M, \mathcal{F}^{r,p}) \mid \bar{\partial}\varphi = 0\}}{\bar{\partial} H^0(M, \mathcal{F}^{r,p-1})}$$

for $p \geq 1$. *Further*

$$H^p(M, \Omega^r) = 0$$

for $p \geq n+1$.

We note that in case N is a Stein manifold, by the result of B. Jennane [2], $H^p(M, \Omega^r) = 0$ for $p \geq 2$.

2. We shall apply the result of 1 to the calculation of $\bar{\partial}$ cohomology of toroidal groups which was partly shown in [3] and [5].

Let G be a toroidal group of complex dimension n, that is, G is a connected complex Lie group without nonconstant global holomorphic functions. Then we may assume $G = \boldsymbol{C}^n / \Gamma$, where Γ is a discrete lattice of $\boldsymbol{C}^n$ generated by $\boldsymbol{R}$-linearly independent vectors $\{e_1, e_2, \cdots, e_n, v_1 = (v_{11}, \cdots, v_{1n}),$ $v_2 = (v_{21}, \cdots, v_{2n}), \cdots, v_q = (v_{q1}, \cdots v_{qn})\}$ over $\boldsymbol{Z}$ and e_i denotes the i-th unit vector of $\boldsymbol{C}^n$. Moreover we may assume det $[\Im u_{ij}; 1 \leq i, j \leq q] \neq 0$, where $\Im v_{ij}$ is the imaginary part of v_{ij}. We denote by π the projection $\boldsymbol{C}^n \ni (z^1, \cdots, z^n) \mapsto (z^1, \cdots, z^q) \in \boldsymbol{C}^q$. Since $\pi(e_i)$, $\pi(v_i)$ $(1 \leq i \leq q)$ are $\boldsymbol{R}$-linearly independent π induces the $\boldsymbol{C}^{*n-q}$-principal bundle

$$\pi : \boldsymbol{C}^n / \Gamma \ni z + \Gamma \mapsto \pi(z) + \Gamma^* \in \boldsymbol{T}^q_{\boldsymbol{C}} := \boldsymbol{C}^q / \Gamma^*$$

over the complex q dimensional torus $\boldsymbol{T}^q_{\boldsymbol{C}}$, where $\Gamma^* := \pi(\Gamma)$. Then, the sheaves $\mathcal{F}^{r,p}$ on $\boldsymbol{C}^n/\Gamma$ over $\boldsymbol{T}^q_{\boldsymbol{C}}$ are defined for $0 \leq r \leq n$ and $0 \leq p \leq q$. We have $H^p(\boldsymbol{C}^n/\Gamma, \Omega^r) \cong \{\varphi \in H^0(\boldsymbol{C}^n/\Gamma, \mathcal{F}^{r,p}) \mid \bar{\partial}\varphi = 0\} / \bar{\partial} H^0(\boldsymbol{C}^n/\Gamma, \mathcal{F}^{r,p-1})$ for $p \geq 1$. $(z^1, \cdots, z^n)$ defines a local coordinate in $\boldsymbol{C}^n/\Gamma$ and

$$f \in H^0(\boldsymbol{C}^n/\Gamma, \mathcal{F}) \text{ means } \frac{\partial f}{\partial \bar{z}^i} = 0 \quad \text{for } q+1 \leq i \leq n. \tag{2.1}$$

We put $v_i := \sqrt{-1}\, e_i$ for $q+1 \leq i \leq n$ and $(z^1, \cdots, z^n) = \sum_{i=1}^n (t^i e_i + t^{n+i} v_i)$. The isomorphism $\boldsymbol{C}^n \ni (z^1, \cdots, z^n) \mapsto (t^1, \cdots, t^{2n}) \in \boldsymbol{R}^{2n}$ induces the isomorphism $\boldsymbol{C}^n/\Gamma \cong \boldsymbol{T}^{n+q} \times \boldsymbol{R}^{n-q}$ as a real Lie group, where $\boldsymbol{T}^{n+q}$ is a real $(n+q)$ dimensional real torus. For $1 \leq i_1 < \cdots < i_r \leq n$ and $1 \leq j_1 < \cdots < j_p \leq q$, put $I := (i_1, \cdots, i_r)$ and $J := (j_1, \cdots, j_p)$, respectively. Let $\varphi = \sum_{I,J} \varphi_{IJ} dz^I \wedge d\bar{z}^J$ $\in H^0(\boldsymbol{C}^n/\Gamma, \mathcal{F}^{r,p})$ be a $\bar{\partial}$-closed form. For each I, $\varphi_I = \sum_J \varphi_{IJ} d\bar{z}^J$ is a $\bar{\partial}$-

closed $(0,p)$-form in C^n/Γ. By the Fourier expansion and (2.1), we can write

$$\varphi_{IJ}=\sum_{m\in Z^{n+q}}\varphi_{IJ}^m(t)$$
$$=\sum_{m\in Z^{n+q}} c_{IJ}^m \exp\left(-2\pi\sum_{i=q+1}^{n} m_i t^{n+i}\right)\exp(2\pi\sqrt{-1}\langle m,t'\rangle)$$

and $\varphi_I=\sum_{m\in Z^{n+q}}\varphi_I^m=\sum_{m\in Z^{n+q}}\sum_J \varphi_{IJ}^m d\bar{z}^J$, where c_{IJ}^m are constant and $\langle m,t'\rangle :=\sum_{i=1}^{n+q} m_i t^i$. From the similar argument to [3], we have the constant from $\sum_J c_{IJ}^0 d\bar{z}^J$ and $\psi_I^m=\sum_{J'}\psi_{IJ'}^m d\bar{z}^{J'}\in H^0(C^n/\Gamma,\mathcal{E}^{0,p-1})$, where $J'=(j_1,\cdots,j_{p-1})$ for each $m\in Z^{n+q}\backslash\{0\}$ such that $\varphi_I^m=\bar{\partial}\psi_I^m$ and $\varphi_I=\sum_J c_{IJ}^0 d\bar{z}^J+\sum_{m\in Z^{n+q}}\bar{\partial}\psi_I^m$. Put $\psi^m:=(-1)^r\sum_{I,J'}\psi_{IJ'}^m dz^I\wedge d\bar{z}^{J'}$ for each $m\in Z^{n+q}\backslash\{0\}$, we have

$$\varphi=\sum_{I,J} c_{IJ}^0 dz^I\wedge d\bar{z}^J+\sum_{m\in Z^{n+q}\backslash\{0\}}\bar{\partial}\psi^m.$$

$\psi:=\sum_{m\in Z^{n+q}\backslash\{0\}}\psi^m$ is a formal solution of $\bar{\partial}$-problem for the $\bar{\partial}$-closed form $\varphi-\sum_{I,J}c_{IJ}^0 dz^I\wedge d\bar{z}^J$. Then similarly to [3] and [4], we obtain

Theorem 2.1. *Let C^n/Γ be a toroidal group, where Γ is a discrete lattice of C^n generated by R-linearly independent vectors $\{e_1,e_2,\cdots,e_n,v_1,v_2,\cdots,v_q\}$, $K_{m,i}:=\sum_{j=1}^n v_{ij}m_j-m_{n+i}$ and $K_m:=\max\{|K_{m,i}|;1\le i\le q\}$ for $m\in Z^{n+q}$. Then the following statements* (1), (2), (3), *and* (4) *are equivalent.*

(1) *There exists $a>0$ such that*

$$\sup_{m\neq 0}\exp(-a\|m^*\|)/K_m<\infty,$$

where $\|m^*\|=\max\{|m_i|;1\le i\le n\}$.

(2) $H^p(C^n/\Gamma,\Omega^r)\cong C\{dz^I\wedge d\bar{z}^J|I=(i_1,\cdots,i_r),\quad J=(j_1,\cdots,j_p),\ 1\le i_1<\cdots<i_r\le n,\ \text{and}\ 1\le j_1<\cdots<j_p\le q\}$, *for* $p\ge 1$, $r=0,1,\cdots,n$. *Then* $\dim H^p(C^n/\Gamma,\Omega^r)=\binom{n}{r}\binom{q}{p}$.

(3) $H^p(C^n/\Gamma,\Omega^r)$ *are Hausdorff locally convex spaces.*

(4) $\bar{\partial}H^0(C^n/\Gamma,\mathcal{E}^{r,p-1})$ *is a closed subspace of the Frechet space* $H^0(C^n/\Gamma,\mathcal{E}^{r,p})$ *for* $p\ge 1$.

Corollary 2.1. *Every toroidal group C^n/Γ satisfies either of the following statements* (a) *and* (b).

(a) $H^p(C^n/\Gamma,\Omega^r)$ *is finite dimensional for any p and $0\le r\le n$.*

(b) $H^p(C^n/\Gamma,\Omega^r)$ *is a non-Hausdorff locally convex space for $1\le p\le q$ and $0\le r\le n$.*

Remark. C. Vogt [5] showed the equivalence of (1) and (2) in the above theorem by Dolbeault theory. In the previous paper [3] we obtained the above theorem and corollary in case $r=0$.

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