76. On a Theorem of Landau. II

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§ 1. **Introduction.** Let $\rho=\beta+i\gamma$ run over the non-trivial zeros of the Riemann zeta function $\zeta(s)$. Landau [9] showed that for fixed $x>1$

$$\sum_{0<\gamma\le T} x^{\rho}=-\frac{T}{2\pi}\Lambda(x)+O(\log T),$$

where we put $\Lambda(x)=\log p$ if $x=p^k$ with a prime number p and a positive integer k, and $=0$ otherwise. In [4], the author has refined this under the Riemann Hypothesis (R.H.) as follows.

$$\sum_{0<\gamma\le T} x^{\frac{1}{2}+i\gamma}=-\frac{T}{2\pi}\Lambda(x)+\frac{x^{\frac{1}{2}+iT}\log(T/2\pi)}{2\pi i\log x}+O\Big(\frac{\log T}{\log\log T}\Big).$$

Here we shall refine this further. It T is not the ordinate of a zero of $\zeta(s)$, let $S(T)$ denote the value of

$$\frac{1}{\pi}\arg\zeta\Big(\frac{1}{2}+iT\Big)$$

obtained by continuous variation along the straight lines joining 2, $2+iT$, $\frac{1}{2}+iT$, starting with the value 0. If T is the ordinate of a zero, let $S(T)=S(T+0)$. We shall prove the following theorem.

Theorem 1 (Under R.H.). *For fixed $x>1$ and $T>T_0$, we have*

$$\sum_{0<\gamma\le T} x^{i\gamma}=-\frac{T}{2\pi}\frac{\Lambda(x)}{\sqrt{x}}+\frac{x^{iT}\log(T/2\pi)}{2\pi i\log x}+x^{iT}S(T)+O\Big(\frac{\log T}{(\log\log T)^2}\Big).$$

We know that $S(T)\ll(\log T/\log\log T)$ under R.H. and that the normal order of $S(T)$ is $(1/2\pi)\sqrt{\log\log T}$. Hence the third term of the right hand side in the above formula might be reduced in the remainder term.

The dependence on x in Landau's theorem is also important and has been studied by Gonek [7], [8] and Fujii [3], for example. Here we shall refine, under R.H., our previous results in [3] as follows.

Theorem 2 (Under R.H.). *For $x>1$ and $T>T_0$, we have*

$$\begin{aligned}\sum_{0<\gamma\le T} x^{i\gamma}=&-\frac{T}{2\pi}\frac{\Lambda(x)}{\sqrt{x}}+M(x,T)+x^{iT}S(T)+O(B(x,T))\\&+O\Big(\mathrm{Min}\Big\{\sqrt{x}\log x\cdot\frac{\log T}{(\log\log T)^2},\ \sqrt{x}\log(2x)\\&+x^{1/\log\log T}\frac{\log T}{\log((\log T/x)+2)}\\&+\sqrt{x}\sqrt{\frac{\log T}{\log\log T}}\frac{1}{\log((x/\log T\cdot\log\log T)+2)}\Big\}\Big),\end{aligned}$$

where

$$M(x,T)\equiv\frac{1}{2\pi}\int_1^T x^{it}\log\Big(\frac{t}{2\pi}\Big)dt$$
$$=\begin{cases}\dfrac{x^{iT}\log(T/2\pi)}{2\pi i\log x}+O\Big(\dfrac{1}{\log x}+\dfrac{1}{\log^2 x}\Big) & \text{if } \dfrac{1}{\log T}\ll\log x\\ O\Big(\mathrm{Min}\Big(\dfrac{\log T}{\log x},\, T\log T\Big)\Big) & \text{if } \log x\ll\dfrac{1}{\log T}\end{cases}$$

and

$$B(x,T)\equiv\frac{1}{\sqrt{x}}\sum_{\substack{(x/2)<k<2x\\ k\neq x}}\Lambda(k)\,\mathrm{Min}\Big(T,\frac{1}{|\log(x/k)|}\Big)$$
$$=O\Big(\frac{\log(2x)}{\sqrt{x}}\mathrm{Min}\Big(T,\frac{x}{\langle x\rangle}\Big)\Big)+O(\sqrt{x}\log(3x)\log\log(3x)),$$

$\langle x\rangle$ being the distance from x to the nearest prime power other than x itself.

In some problems it is desirable to get an evaluation of the above sum without using any unproved hypothesis. Here we notice the following theorem.

Theorem 3. *Suppose that for $\sigma\geq\frac{1}{2}$, a positive constant θ satisfies*

$$|\{\rho=\beta+i\gamma\,;\,0<\gamma<T,\,\beta>\sigma\}|\ll T\log T\cdot e^{-(\sigma-\frac{1}{2})\theta\log T}.$$

Then for $1<x\ll T^{\mathrm{Min}(2,\theta)-\varepsilon}$ and $\varepsilon>0$,

$$\sum_{0<\gamma\leq T}x^{i\gamma}\ll T\log x+\mathrm{Min}\Big(\frac{\log T}{\log x},\,T\log T\Big).$$

We may take $\theta=\frac{8}{7}-\varepsilon$, $\varepsilon>0$, by Conrey's improvement [1] of Selberg's density theorem in [10].

We shall prove Theorems 1 and 2 using our previous arguments in [3]. The present improvement comes mainly from the following theorem which is an improvement of p. 529 of [2].

Theorem 4 (Under R.H.). *For $T>T_0$,*

$$\int_{\frac{1}{2}}^{2}|\log\zeta(\sigma+iT)|\,d\sigma\ll\frac{\log T}{(\log\log T)^2}.$$

§2. Proof of Theorem 4. We assume the Riemann Hypothesis in this section. We put $Y=\log T$ and $\sigma_1=\frac{1}{2}+\frac{1}{\log Y}$. We notice first that

$$\int_{1/2}^{\sigma_1}|\log\zeta(\sigma+iT)|\,d\sigma$$
$$=\int_{1/2}^{\sigma_1}\Big|A\frac{\log T}{\log\log T}-\log|\zeta(\sigma+iT)|-i\arg\zeta(\sigma+iT)-A\frac{\log T}{\log\log T}\Big|\,d\sigma$$
$$\leq\int_{1/2}^{\sigma_1}\Big(A\frac{\log T}{\log\log T}-\log|\zeta(\sigma+iT)|\Big)d\sigma$$
$$+\int_{1/2}^{\sigma_1}|\arg\zeta(\sigma+iT)|\,d\sigma+A\frac{\log T}{(\log\log T)^2},$$

since it is known (cf. p. 300 of Titchmarsh [11]) that with a positive constant A,

$$\log|\zeta(\sigma+iT)|\le A\frac{\log T}{\log\log T}\qquad \text{for } \frac{1}{2}\le\sigma\le\sigma_1.$$

Since, by 14.13.6 and 14.14.3 of Titchmarsh [11],

$$\int_{1/2}^{\sigma_1}\log|\zeta(\sigma+iT)|\,d\sigma\ll\frac{\log T}{(\log\log T)^2}$$

and

$$\arg\zeta(\sigma+iT)\ll\frac{\log T}{\log\log T}\qquad \text{for } \frac{1}{2}\le\sigma\le\sigma_1,$$

we see that

$$\int_{1/2}^{\sigma_1}|\log\zeta(\sigma+iT)|\,d\sigma\ll\frac{\log T}{(\log\log T)^2}.$$

We next treat the integral over the interval $\sigma_1\le\sigma\le 2$. Applying Selberg's expression (cf. 14.21.4 of [11])

$$\frac{\zeta'}{\zeta}(\sigma+iT)=-\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma+iT}}+O\left(Y^{(1/2)-\sigma}\left|\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1+iT}}\right|\right)+O(Y^{(1/2)-\sigma}\log T),$$

we get first

$$\begin{aligned}\log\zeta(\sigma+iT)&=-\int_{\sigma}^{2}\frac{\zeta'}{\zeta}(\sigma+iT)\,d\sigma+\log\zeta(2+iT)\\&=\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma+iT}\log n}+O\left(\frac{Y^{(1/2)-\sigma}}{\log Y}\left|\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1+iT}}\right|\right)+O\left(\frac{Y^{(1/2)-\sigma}}{\log Y}\log T\right)+O(1),\end{aligned}$$

where we put

$$\Lambda_Y(n)=\begin{cases}\Lambda(n) & \text{for } 1\le n\le Y\\ \Lambda(n)\dfrac{\log(Y^2/n)}{\log Y} & \text{for } Y\le n\le Y^2.\end{cases}$$

Using this we get

$$\begin{aligned}\int_{\sigma_1}^{2}|\log\zeta(\sigma+iT)|\,d\sigma&\ll\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1}\log^2 n}+\frac{1}{\log^2 Y}\sum_{n<Y^2}\frac{\Lambda_Y(n)}{n^{\sigma_1}}+\frac{\log T}{\log^2 Y}\\&\ll\sum_{n<Y^2}\frac{\Lambda(n)}{\sqrt{n}\log^2 n}+\frac{\log T}{\log^2 Y}\ll\frac{\log T}{(\log\log T)^2}.\end{aligned}$$

Combining the above two estimates, we get our theorem.

§3. Proof of Theorems 1 and 2. We assume the Riemann Hypothesis in this section. We shall follow the arguments in pp. 52–54 of [3] and omit some of the details.

$$\begin{aligned}\sum_{0<\gamma\le T}x^{i\gamma}&=M(x,T)-i\log x\cdot\int_C^T\cos(t\log x)S(t)\,dt\\&\quad+\log x\cdot\int_C^T\sin(t\log x)S(t)\,dt+x^{iT}S(T)+O(1),\end{aligned}$$

where C is some positive constant.

We put $\delta=\dfrac{1}{\log(9x)}$. Then we get

$$\int_C^T \cos(t\log x)S(t)\,dt$$
$$=\Im\left\{\frac{1}{\pi i}\left(\int_{1+\delta+iC}^{1+\delta+iT}-\int_{(1/2)+iT}^{1+\delta+iT}+\int_{(1/2)+iC}^{1+\delta+iC}\right)\cos\left(-i\left(z-\frac{1}{2}\right)\log x\right)\log\zeta(z)\,dz\right)\right\}$$
$$=\Im\left\{\frac{1}{\pi i}(S_1+S_2+S_3)\right\}, \text{ say.}$$

A direct application of the above Theorem 4 yields

$$\log x\cdot S_2\ll\sqrt{x}\log x\cdot\int_{1/2}^{1+\delta}|\log\zeta(\sigma+iT)|\,d\sigma\ll\sqrt{x}\log x\cdot\frac{\log T}{(\log\log T)^2}.$$

We shall improve the dependence on x a little bit as follows. We put

$$W=\log T \text{ if } x<\frac{1}{2}\log T \text{ and} =\log T\log\log T \text{ if } x\geq\frac{1}{2}\log T.$$

We put further $\sigma_1=\dfrac{1}{2}+\dfrac{1}{\log W}$. Then

$$\log x\cdot S_2\ll\log x\cdot\left|\int_{1/2}^{1+\delta}x^{\sigma-(1/2)}\log\zeta(\sigma+iT)\,d\sigma\right|+\log x\cdot\frac{\log T}{(\log\log T)^2}$$
$$\ll\sqrt{x}\,|\log\zeta(1+\delta+iT)|+x^{\sigma_1-(1/2)}|\log\zeta(\sigma_1+iT)|$$
$$+x^{\sigma_1-(1/2)}\cdot\log x\cdot\frac{\log T}{(\log\log T)^2}+\left|\int_{\sigma_1}^{1+\delta}x^{\sigma-(1/2)}\frac{\zeta'}{\zeta}(\sigma+iT)\,d\sigma\right|$$
$$\ll\sqrt{x}\log\log(3x)+x^{\sigma_1-(1/2)}\left(\frac{\log T}{\log\log T}+\log x\cdot\frac{\log T}{(\log\log T)^2}\right)$$
$$+\left|\int_{\sigma_1}^{1+\delta}x^{\sigma-(1/2)}\frac{\zeta'}{\zeta}(\sigma+iT)\,d\sigma\right|.$$

Using Selberg's expression of $\dfrac{\zeta'}{\zeta}(\sigma+iT)$ as used in the previous section, the last integral is

$$=-\int_{\sigma_1}^{1+\delta}x^{\sigma-(1/2)}\sum_{n<W^2}\frac{\Lambda_W(n)}{n^{\sigma+iT}}\,d\sigma+O\left(\left|\sum_{n<W^2}\frac{\Lambda_W(n)}{n^{\sigma_1+iT}}\right|\int_{\sigma_1}^{1+\delta}x^{\sigma-(1/2)}W^{(1/2)-\sigma}d\sigma\right)$$
$$+O\left(\log T\cdot\int_{\sigma_1}^{1+\delta}x^{\sigma-(1/2)}W^{(1/2)-\sigma}d\sigma\right)$$
$$\ll\sum_{n<W^2}\Lambda_W(n)x^{-(1/2)}\int_{\sigma_1}^{1+\delta}\left(\frac{x}{n}\right)^{\sigma}d\sigma+\sum_{n<W^2}\frac{\Lambda_W(n)}{n^{\sigma_1}}\sqrt{\frac{W}{x}}\int_{\sigma_1}^{1+\delta}\left(\frac{x}{W}\right)^{\sigma}d\sigma$$
$$+\log T\cdot\sqrt{\frac{W}{x}}\int_{\sigma_1}^{1+\delta}\left(\frac{x}{W}\right)^{\sigma}d\sigma$$
$$=\Sigma_1+\Sigma_2+\Sigma_3, \text{ say.}$$

$$\Sigma_1\ll\sqrt{x}\log(3x)+x^{\sigma_1-(1/2)}\left\{\sum_{n<W}\frac{\Lambda(n)}{n^{\sigma_1}}+\sum_{W<n\leq W^2}\frac{\Lambda(n)}{n^{\sigma_1}}\frac{\log(W^2/n)}{\log W}\right\}$$
$$\ll\sqrt{x}\log(3x)+x^{\sigma_1-(1/2)}\frac{W}{\log W}.$$

$$\Sigma_2+\Sigma_3\ll\begin{cases}\dfrac{\sqrt{x}\,W^{(1/2)-\delta}}{\log W\cdot\log(x/W)}+\log T\sqrt{x}\cdot\dfrac{W^{-(1/2)-\delta}}{\log(x/W)} & \text{if } x\geq 2W\\[2ex] \dfrac{W}{\log W}+\log T & \text{if } \dfrac{W}{2}\leq x\leq 2W\\[2ex] \dfrac{x^{\sigma_1-(1/2)}W}{\log W\cdot\log(W/x)}+\dfrac{x^{\sigma_1-(1/2)}\log T}{\log(W/x)} & \text{if } x\leq\dfrac{1}{2}W.\end{cases}$$

Hence, we get for $x>1$,

$$\log x\cdot S_2\ll \mathrm{Min}\Big\{\sqrt{x}\log x\cdot\frac{\log T}{(\log\log T)^2},\ \sqrt{x}\log(2x)+x^{1/\log\log T}$$
$$\times\frac{\log T}{\log((\log T/x)+2)}+\sqrt{x}\sqrt{\frac{\log T}{\log\log T}}\frac{1}{\log((x/\log T\cdot\log\log T)+2)}\Big\}.$$

We get the same upper bound for $\log x\cdot S_3$.

As in p. 54 of [3], we get

$$\log x\cdot S_1=\frac{i}{2}T\frac{\Lambda(x)}{\sqrt{x}}+O(B(x,T))+O(\sqrt{x}\log\log(3x)),$$

where $B(x,T)$ is defined in the statement of Theorem 2.

In a similar manner, we can treat the integral $\int_C^T\sin(t\log x)S(t)\,dt$ and get our assertions as described in Theorem 2, and hence those in Theorem 1.

§4. **Proof of Theorem 3.** We do not assume R.H. in this section. As in the previous section, we get

$$\sum_{0<\gamma\le T}x^{i\gamma}=M(x,T)-i\log x\cdot\int_C^T\cos(t\log x)S(t)\,dt$$
$$+\log x\cdot\int_C^T\sin(t\log x)S(t)\,dt+O(\log T)$$
$$=M(x,T)-i\log x\cdot U_1+\log x\cdot U_2+O(\log T),\ \text{say}.$$

We put $\delta=\dfrac{1}{\log(9x)}$. Then as in p. 104 of [4], we get

$$U_1=\Im\Big(\frac{1}{\pi i}\Big(\Big(\int_{1+\delta+iC}^{1+\delta+iT}-\int_{(1/2)+iT}^{1+\delta+iT}+\int_{(1/2)+iC}^{1+\delta+iC}\Big)\cos\Big(-i\Big(z-\frac{1}{2}\Big)\log x\Big)\log\zeta(z)\,dz+R_1\Big)\Big)$$
$$=\Im\Big(\frac{1}{\pi i}(U_3+U_4+U_5+R_1)\Big),\ \text{say},$$

where we put

$$R_1=2\pi i\sum_{\beta>(1/2),0<\gamma<T}\int_{(1/2)+i\gamma}^{\beta+i\gamma}\cos\Big(-i\Big(z-\frac{1}{2}\Big)\log x\Big)dz.$$

By our assumption, we get

$$R_1\ll\sum_{\beta>(1/2),0<\gamma<T}\int_{1/2}^{\beta}x^{\sigma-(1/2)}d\sigma\ll\int_{1/2}^{1}\sum_{\beta>\sigma,0<\gamma<T}x^{\sigma-(1/2)}d\sigma$$
$$\ll T\log T\int_{1/2}^{1}e^{-\theta(\sigma-(1/2))\log T+(\sigma-(1/2))\log x}d\sigma\ll T.$$
$$U_4\ll\sqrt{x}\int_{1/2}^{1+\delta}|\log\zeta(\sigma+iT)|\,d\sigma\ll\sqrt{x}\log T.$$
$$U_5\ll\sqrt{x}.$$
$$U_3=\frac{i}{2}x^{(1/2)+\delta}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{1+\delta}\log n}\int_C^T\Big(\frac{x}{n}\Big)^{it}dt+\frac{i}{2}x^{-((1/2)+\delta)}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^{1+\delta}\log n}\int_C^T\Big(\frac{1}{nx}\Big)^{it}dt$$
$$\ll T\frac{\Lambda(x)}{\sqrt{x}\log x}+\frac{\sqrt{x}\log\log(3x)}{\log(2x)}+\frac{1}{\log(2x)}B(x,T),$$

where $B(x,T)$ is the same as above.

Thus we get

$$\log x\cdot U_1 \ll T\log x+\sqrt{x}\log x\log T+B(x,T).$$

U_2 can be estimated similarly. Since

$$B(x,T)\ll T\log x+1,$$

we get our assertion as stated in Theorem 3.

§5. Concluding remarks. 5-1. Since

$$x^{-(1/2)}\sum_{\gamma\le T}x^{\rho}-\sum_{\gamma\le T}x^{i\gamma}=\log x\int_{1/2}^{1}\Big(\sum_{\gamma\le T,\beta>\sigma}x^{i\gamma}\Big)x^{\sigma-(1/2)}d\sigma\ll T\log x,$$

we get another proof of Theorem 3, by applying Gonek's estimate on $\sum_{\gamma\le T}x^{\rho}$ in [7] and [8].

5-2. Some of the theorems announced in [5] can be improved since we have used the arguments in [3], which are now improved. For example, Theorem 3 of [5] for a fixed x can be improved as follows.

Theorem (Under R.H.). *For any $x>1$ and $T>T_0$, we have*

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma+\gamma'\le T}}x^{i(\gamma+\gamma')}=\frac{1}{8\pi^2}\frac{\Lambda^2(x)}{x}T^2+\frac{x^{iT}\,T\log^2 T}{4\pi^2 i\log x}+O\Big(\frac{T\log^2 T}{(\log\log T)^2}\Big),$$

where γ and γ' run over the imaginary parts of the zeros of $\zeta(s)$.

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