8. A Counterexample in the Theory of Prehomogeneous Vector Spaces

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1. Let G be a linear algebraic group defined over the complex number field C, (G, ρ, V) a prehomogeneous vector space and Ω the open dense G-orbit in V. (See below for the definitions.) If G is reductive and (G, ρ, V) is regular, then the open subvariety Ω of V is an affine variety [2]. Here the regularity condition is known to be essential, but it seems that the reductivity of G was expected not to be essential. The purpose of this note is to give a counterexample to this expectation.

2. **Prehomogeneous vector spaces.** Let G be as above, $V=C^k$, and $\rho: G\to GL(V)$ a rational representation of G. Such a triplet (G, ρ, V) is called a *prehomogeneous vector space* if V has an open dense G-orbit. (Here and below, we exclusively consider the Zariski topology.) Such an orbit is unique and we shall denote it by Ω. A prehomogeneous vector space (G, ρ, V) is called *regular* if there exists a polynomial function $f(x)=f(x_1, \cdots, x_k)$ on V which satisfies the following two conditions:

(R1) There exists a rational character ϕ of G such that $f(\rho(g)v)=\phi(g)f(v)$ for any $g\in G$ and $v \in V$.

(R2) $$\det\left(\frac{\partial^2 \log f}{\partial x_i \partial x_j}\right)_{1\le i,j\le k}\neq 0 \qquad \text{on } \Omega.$$

3. **Tits system.** Let $G=GL_n(C)$, B be the Borel subgroup of G consisting of upper triangular matrices, T the maximal torus of B consisting of diagonal matrices, $N=N_G(T)$ the normalizer of T in G and $W=N/T$ the Weyl group. Let $\mathfrak{S}_n$ be the symmetric group acting on $\{1, 2, \cdots, n\}$ and $\dot{W}$ the group of permutation matrices in $GL_n(C)$. Then we have natural isomorphisms $\mathfrak{S}_n\simeq \dot{W}\simeq W$, by which we shall identify these three groups. Let S be the set of transpositions $\{(1,2), (2,3), \cdots, (n-1, n)\}$ and

$$w_0=\begin{pmatrix}1 & 2 & \cdots & n\\ n & n-1 & \cdots & 1\end{pmatrix}.$$

Then (G, B, N, S) is a Tits system [1; chapter 4, section 2]. For a subset X of S, let W_X be the subgroup of W generated by X and $G_X=BW_XB$. Every element $w\in W$ can be expressed as $w=s_1s_2\cdots s_a$ $(s_i\in S)$. Define the length $l(w)$ of w to be the minimum of the length a of such an expression. It is known that $l(w)=\dim BwB-\dim B$. If $x, y\in W$ can be expressed as

$$x=s_1s_2\cdots s_a \quad (s_i\in S, a=l(x)),$$

and

$$y=s_{i_1}s_{i_2}\cdots s_{i_b} \quad (1\le i_1<i_2<\cdots i_b\le a),$$

then we write $x \leq y$. This relation defines a partial order in W which is called the *Bruhat order*. It is known that $x \leq y$ if and only if $\overline{BxB} \subset \overline{ByB}$, where the closure may be taken in $GL_n(C)$ or $M_n(C)$.

4. **Counterexample.** Let X and Y be two subsets of S. Define a $G_X \times G_Y$-action on $M_n(C)$ (the totality of $n \times n$-matrices) by $\rho(g_1, g_2)v = g_1 v g_2^{-1}$ for $v \in M_n(C)$ and $(g_1, g_2) \in G_X \times G_Y$.

(1) $(G_X \times G_Y, \rho, M_n(C))$ is a prehomogeneous vector space, whose open dense orbit is $G_X w_0 G_Y$.

(2) Moreover, it is regular, since $f(v) = \det v$ $(v \in M_n(C))$ satisfies the conditions (R1) and (R2).

(3) Let $\{w_1, \cdots, w_m\}$ be the set of maximal elements of $W - W_X w_0 W_Y$ with respect to the Bruhat order. Then the irreducible components of $M_n(C) - G_X w_0 G_Y$ are

$$C_0 = \{v \in M_n(C) \mid \det v = 0\}$$

and

$$C_i = \text{closure of } Bw_iB \text{ in } M_n(C) \quad (1 \leq i \leq m).$$

In fact

$$\begin{aligned} M_n(C) - G_X w_0 G_Y &= C_0 \cup (G - G_X w_0 G_Y) \\ &= C_0 \cup B(W - W_X w_0 W_Y)B = \bigcup_{0 \leq i \leq m} C_i. \end{aligned}$$

(4) An open subvariety O of C^k is an affine variety if and only if every irreducible component of $C^k - O$ is a hypersurface. In fact, a regular function outside of a subvariety Z of codimension ≥ 2 extends to the whole space. Hence the spectrum of the ring of regular functions contains Z.

(5) The following conditions are equivalent:

(i) The open orbit $G_X w_0 G_Y$ is an affine variety.

(ii) $\dim C_i = \dim G - 1$ $(1 \leq i \leq m)$.

(iii) $l(w_i) = l(w_0) - 1$ $(1 \leq i \leq m)$.

Now we can give a counterexample. Let $n=3$, $s_1 = (1, 2)$, $s_2 = (2, 3)$ and $X = Y = s_1$. Then $w_0 = s_1 s_2 s_1$ and $W - W_X w_0 W_Y = \{1, s_1\}$. Hence $m=1$ and $w_1 = s_1$. Since $l(w_0) = 3$ and $l(w_1) = 1$, $G_X w_0 G_Y$ is not an affine variety.

References

[1] N. Bourbaki: Groupes et Algèbres de Lie. Chapitres 4, 5 et 6, Hermann, Paris (1968).

[2] M. Sato and T. Kimura: A classification of irreducible prehomogeneous vector spaces and their relative invariants. Nagoya Math. J., **65**, 1–155 (1977).