63. *q*-analogue of de Rham Cohomology Associated with Jackson Integrals. II

By Kazuhiko AOMOTO
Department of Mathematics, Nagoya University

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We follow the same terminologies as in Part I (see [3]).

1. *Critical points and corresponding stable q-cycles.* We assume for simplicity that q is real such that $0<q<1$. We put $\alpha=N\eta+\alpha'$ and study the asymptotic behaviour of Jackson integrals $\langle\varphi\rangle, \varphi\in V$, for $N\to+\infty, \eta\in\check{X}$ and $\alpha'\in \boldsymbol{C}^n$ being fixed. Since $\Phi(t)=(t^\eta)^N\cdot t^{\alpha'}\cdot\Pi_{j=1}^m\frac{(a_j't^{\mu_j})_\infty}{(a_jt^{\mu_j})_\infty}$, the major part of $|\Phi|$ is played by the absolute value $|t^\eta|$ for $N\to+\infty$. $|t^\eta|$ attains a maximum if and only if the level function $L_\eta(\log_q t)=\mathrm{Re}\,(\eta, \log_q t)$ is a minimum, where (η,λ) denotes $\eta(\lambda)$ for $\lambda\in X_C=X\otimes \boldsymbol{C}$.

We are going to search for points $t=q^\lambda$ in $\bar{X}$ for $\lambda \bmod \frac{2\pi i}{\log q}X$ satisfying the following 2 properties:

(i) $\log_q a_j'+(\mu_j,\lambda)\equiv 1,2,3,\cdots \bmod \frac{2\pi i}{\log q}\boldsymbol{Z}$, *for* $j\in J$, J *being a set of* n *arguments in* $\{\pm1,\pm2,\cdots,\pm m\}$ *such that* μ_j, $j\in J$, *are linearly independent in* $\check{X}_{\boldsymbol{R}}=\check{X}\otimes\boldsymbol{R}$. *We denote by* $\bar{X}_J$ *the countable set in* $\bar{X}$ *consisting of these points t.*

(ii) $L_\eta(\lambda)$ *attains a minimum on the subset of* $\bar{X}_J$ *consisting of the points which are X-equivalent to* $\lambda \bmod \frac{2\pi i}{\log q}X$.

This is a very special case of linear programming problem investigated in [6] or [11].

We say that a point $t=q^\lambda$ satisfying (i) and (ii) is a *critical point with respect to the level function* $L_\eta(\lambda)$. We denote by $Cr_J=Cr_J(L_\eta)$ the set of all critical points in $\bar{X}_J$ and by $Cr(L_\eta)$ the union $\bigcup_J Cr_J(L_\eta)$.

Now we make the following assumptions of genericity.

Assumption 1. *For each J, the set* Cr_J *is finite or empty. We denote by* κ_J *its number*:

$$Cr_J(L_\eta)=\{\xi_J^{(1)},\cdots,\xi_J^{(\kappa_J)}\}. \tag{1.1}$$

Assume that $L_\eta(\xi_J^{(r)})\neq L_\eta(\xi_J^{(s)})$ *for every pair* $r,s,r\neq s$. *Then* κ_J *turns out equal to* $[\mu_{j_1},\cdots,\mu_{j_n}]^2$ *or* 0. *We say that J is stable if* $\kappa_J>0$.

Assumption 2. $a_k'\xi^{\mu_k}\neq 1, q^{\pm1}, q^{\pm2},\cdots$ *for any* $k\in\{\pm1,\cdots,\pm m\}-J$ *and* $\xi\in Cr_J(L_\eta)$.

From these assumptions we see that, for each $J=\{j_1,\cdots,\pm j_n\}$,

$J\subset\{\pm 1, \cdots, m\}$, the only one choice of signs $\{\varepsilon_1 j_1, \cdots, \varepsilon_n j_n\}$ is stable for $\varepsilon_\nu=\pm 1$. This occurs if and only if

(1.2) $$\varepsilon_1\varepsilon_2\cdots\varepsilon_n[\mu_{j_1}, \cdots, \mu_{j_n}]>0, \quad \text{and}$$

(1.3) $$[\eta, \varepsilon_1\mu_{j_1}, \cdots, \varepsilon_{\nu-1}\mu_{j_{\nu-1}}, \varepsilon_{\nu+1}\mu_{j_{\nu+1}}, \cdots, \varepsilon_n\mu_{j_n}](-1)^{\nu-1}>0$$

for all ν. Hence the total number of critical points $\kappa=\sharp|Cr(L_\eta)|$ is given by

(1.4) $$\kappa=\sum_J [\mu_{j_1}, \cdots, \mu_{j_n}]^2=\det\left(\left(\sum_{j=1}^m \mu_j(\chi_r)\mu_j(\chi_s)\right)\right)_{1\le r,s\le n}.$$

Under the above 2 assumptions, deduce the crucial

Lemma 1.1. *A point $t=q^\lambda$ in the algebraic torus $\bar{X}$ is critical for $L_\eta(\lambda)$ if and only if*

(1.5) $$b_\chi^-(q^{\lambda-\chi})=0$$

for any $\chi\in X$ such that $(\eta, \chi)>0$.

In fact, suppose that q^λ is an element of $Cr_J(L_\eta)$. Then for each $\chi\in X$ such that $(\eta,\chi)>0$, we may assume that there exist a non-empty subset K of J such that $\mu_k(\chi)>0$ for $k\in K$ and $\mu_k(\chi)\le 0$ for $k\in J-K$. By Assumptions 1 and 2, we see that one of $(a'_k t^{\mu_k} q^{-\mu_k(\chi)})_{\mu_k(\chi)}$ vanishes for $k\in K$. Hence $b_\chi^-(^{\lambda-\chi})$ vanishes. Conversely, suppose (1.5) holds for all χ such that $(\eta,\chi)>0$. Then by Assumptions 1 and 2, one can find a subset $J=\{j_1, \cdots, j_n\}\subset\{\pm 1, \cdots, \pm m\}$ such that $q^\lambda\in\bar{X}_J$. If q^λ is not itself critical, then there would exist $\chi\in X$ such that $(\eta,\chi)>0$ and $q^{\lambda-\chi}\in\bar{X}_J$. Hence $b_\chi^-(q^{\lambda-\chi})\neq 0$ which is a contradiction.

Definition 1. We denote by $\mathfrak{c}(\xi)$ the set of all $t=\chi\cdot\xi$, $\chi\in X$, such that $L_\eta(\log_q t)\ge L_\eta(\log_q \xi)$ so that ξ is a minimum point in $\mathfrak{c}(\xi)$. We call such a $\mathfrak{c}(\xi)$ *stable cycle* if ξ is critical. There are κ stable cycles say $\mathfrak{c}(\xi^{(1)}), \cdots, \mathfrak{c}(\xi^{(\kappa)})$.

Since $\xi^{(r)}$ differ from each other, the following holds.

Lemma 1.2. *There exist κ Laurent polynomials $\varphi_r\in\mathcal{L}$, $1\le r\le\kappa$, such that $\varphi_r(\xi^{(s)})=\delta_{r,s}$, $1\le s\le\kappa$.*

Definition 2. Suppose that $\mathfrak{c}(\xi^{(s)})$ is determined by $J\subset\{\pm 1, \cdots, \pm m\}$ such that $j_1<0, \cdots, j_l<0$ and $j_{l+1}>0, \cdots, j_n>0$. Then the integration $\int_{\mathfrak{c}(\xi^{(s)})}\Phi\varphi\varpi$, $\varphi\in V$, is generally impossible because Φ has poles on $\mathfrak{c}(\xi^{(s)})$. We must replace Φ by another Φ' after the substitutions T_j in (1.4) of Part I for $j=j_1, \cdots, j_l$ so that $\int_{\mathfrak{c}(\xi^{(s)})}\Phi'\varphi_r\varpi$ is well defined. We shall call this modification the *regularization of integration of Φ* and denote it by $\operatorname{reg}\int_{\mathfrak{c}(\xi^{(s)})}\Phi\varphi_r\varpi$.

Lemma 1.3.

(1.6) $$\det\left(\left(\operatorname{reg}\int_{\mathfrak{c}(\xi^{(s)})}\Phi\varphi_r\varpi\right)\right)_{1\le r,s\le\kappa}\neq 0.$$

This fact follows from the asymptotic behaviours of $\langle\varphi_1\rangle, \cdots, \langle\varphi_\kappa\rangle$. Indeed, we may assume that $j_1>0, \cdots, j_n>0$ for $\mathfrak{c}(\xi^{(s)})$, since the regularized ones are reduced to this case. Then for $N\to+\infty$.

(1.7) $$\int_{\mathfrak{c}(\xi^{(s)})}\Phi\varphi_r\varpi\sim(1-q)^n\delta_{r,s}(\xi^{(s)})^\alpha\Pi_{j=1}^m\frac{(a'_j(\xi^{(s)})^{\mu_j})_\infty}{(a_j(\xi^{(s)})^{\mu_j})_\infty}\cdot\left(1+O\left(\frac{1}{N}\right)\right),$$

where $\Pi_{j=1}^{m}\frac{(a'_j(\xi^{(s)})^{\mu_j})_\infty}{(a_j(\xi^{(s)})^{\mu_j})_\infty}\neq 0$ by Assumption 2. Hence the lemma.

As a result, $\varphi_1,\cdots,\varphi_\kappa$ are linearly independent in $H^n(\Omega^\cdot, \nabla)$, which implies

$$\dim H^n(\Omega^\cdot,\nabla)\geq\kappa. \tag{1.8}$$

2. *Upper estimate of* $\dim H^n(\Omega^\cdot,\nabla)$. By change of basis we may assume that $\mu_j(\chi_r)\geq 0$ for all j and r. In fact there exists a basis $\{\chi_1,\cdots,\chi_n\}$ such that, $\mu_j(\chi_1)>0$ for all j. The $\chi'_1=\chi_1, \chi'_2=l\chi_1+\chi_2,\cdots,\chi'_n=l\chi_1+\chi_n$ form a basis if $l\in \boldsymbol{Z}$ is sufficiently large and $\mu_j(\chi'_1)>0$ for all j and r. *We also assume* $(\eta,\chi_r)>0$ *for all* r.

We take a $\psi\in\mathcal{L}$. Since

$$\nabla^\chi\psi=\psi-u^\chi\frac{b_\chi^+(t)}{b_\chi^-(t)}Q^\chi\psi, \tag{2.1}$$

$\nabla^\chi\psi\in\mathcal{L}$ if $b_\chi^-(t)\,|\,Q^\chi\psi(t)$, i.e. $\psi(t)=(Q^{-\chi}b_\chi^-(t))\cdot\bar\psi$, $\bar\psi\in\mathcal{L}$, we have

$$\nabla^\chi\psi=(Q^{-\chi}b_\chi^-(t))\cdot\bar\psi(t)-u^\chi b_\chi^+(t)\cdot(Q^\chi\bar\psi(t)). \tag{2.2}$$

We denote by $\mathfrak{a}_q(u)$ the subspace of $\mathcal{L}$ consisting of $\nabla^\chi\psi$ of (2.2)

$$\begin{aligned}\mathfrak{a}_q(u)&=\textstyle\sum_{\chi\in X}\{(Q^{-\chi}b_\chi^-(t))-u^\chi b_\chi^+(t)\cdot Q^\chi\}\mathcal{L}\\ &=\textstyle\sum_{\substack{\chi\in Y\\(\eta,\chi)>0}}\{(Q^{-\chi}b_\chi^-(t))-u^\chi b_\chi^+(t)\cdot Q^\chi\}\mathcal{L}\end{aligned} \tag{2.3}$$

where Y denotes the set of corner vectors spanning the fan F^* defined in Part I, [3]. In the same way we define the subspaces $\mathfrak{a}_q(u;L,L')$ for a sequence (L,L') of non-negative integers $(l_1,\cdots,l_n,l'_1,\cdots,l'_n)$ as follows:

$$\begin{aligned}\mathfrak{a}_q(u;L,L')=&\textstyle\sum_{\chi\in Y}[Q_{a'_j}^{l'_j}Q_{a_j}^{-l_j}\{(Q^{-\chi}b_\chi^-(t))-u^\chi b_\chi^+(t)Q^\chi\}]\mathcal{L}\\ &+\textstyle\sum_{l_j>0}(1-a_jq^{-l_j}t^{\mu_j})\mathcal{L}+\sum_{l'_j>0}(1-a'_jq^{l'_j-1}t^{\mu_j})\mathcal{L}.\end{aligned} \tag{2.4}$$

Remark that $\mathfrak{a}_q(u;\{0\},\{0\})$ coincides with $\mathfrak{a}_q(u)$ itself. We define the ideals in $\mathcal{L}$ by taking $u^\chi\to 0$ (i.e. $u\to 0$) for $(\eta,\chi)>0$:

$$\mathfrak{a}_q(0)=\textstyle\sum_{\substack{\chi\in X\\(\eta,\chi)>0}}(Q^{-\chi}b_\chi^-(t))\mathcal{L}=\sum_{\substack{\chi\in Y\\(\eta,\chi)>0}}(Q^{-\chi}b_\chi^-(t))\mathcal{L}, \tag{2.5}$$

$$\begin{aligned}\mathfrak{a}_q(0;L,L')=&\textstyle\sum_{\substack{\chi\in Y\\(\eta,\chi)>0}}(\Pi_{j=1}^{m}Q_{a'_j}^{l'_j}Q_{a_j}^{-l_j})(Q^{-\chi}b_\chi^-(t))\mathcal{L}\\ &+\textstyle\sum_{l'_j>0}(1-a_j'^{\,l'_j-1}t^{\mu_j})\mathcal{L}+\sum_{l_j>0}(1-a_jq^{-l_j}t^{\mu_j})\mathcal{L}.\end{aligned} \tag{2.6}$$

Then $\mathfrak{a}_q(0;L,L')$ is identical with $\mathcal{L}$ itself provided $\sum_{j=1}^{m}(l_j+l'_j)>0$. Furthermore

Lemma 2.1. *There exists a non-zero Laurent polynomial* $G(u|a,a')$ *in* u_j, a_j, a'_j *such that*

$$\Pi_{\chi'}G(uq^{\chi'}|a,a')t^\chi\equiv 0 \bmod \mathfrak{a}_q(u;L,L'),\ \textit{for}\ \chi\in X, \tag{2.7}$$

provided $\sum_{j=1}^{m}(l_j+l'_j)>0$, *where* χ' *moves over the set of all points* $\chi'=\sum_{j=1}^{n}\nu'_j\chi_j$ *such that* $\nu'_1=\nu_1,\cdots,\nu'_{j-1}=\nu_{j-1}$, $0\leq\nu'_j\leq\nu_j$ *or* $\nu_j\leq\nu'_j\leq 0$, *and* $\nu'_{j+1}=\nu_{j+1},\cdots,\nu'_n=\nu_n$.

We now make the assumption

Assumption 3. $G(uq^\chi|a,a')$ *don't vanish for any* $\chi\in X$.

Then from Lemma 2.1, we have

Lemma 2.2. *Under Assumptions* 1–3, $\mathfrak{a}_q(u;L,L')=\mathcal{L}$ *for all* (L,L') *such that* $\sum_{j=1}^{m}(l_j+l'_j)>0$, *whence the morphism*

$$\mathcal{L}/\mathfrak{a}_q(u) \rightarrow H^n(\Omega^\cdot, \nabla) \longrightarrow 0 \tag{2.8}$$

is exact.

In fact every element φ in (1.6) of Part I can be expressed by $\nabla\psi + \varphi'$ for $\psi \in \Omega^{n-1}$ and $\varphi' \in V$, φ' having the same form such that $\sum_{j=1}^m (l_j + l'_j)$ is smaller. By decreasing induction on $\sum_{j=1}^m (l_j + l'_j)$, one deduce the surjectivity (see the reduction argument in [1])

$$\mathcal{L}/\mathcal{L} \cap \nabla\Omega^{n-1} \longrightarrow H^n(\Omega^\cdot, \nabla) \longrightarrow 0. \tag{2.9}$$

The lemma follows from this because of the inclusion $\mathfrak{a}_q(u) \subset \mathcal{L} \cap \nabla\Omega^{n-1}$.

As an immediate consequence we have

Corollary to Lemma 2.2. $\dim \mathcal{L}/\mathfrak{a}_q(u) \geq \dim H^n(\Omega^\cdot, \nabla)$.

Lemma 2.3. $\dim \mathcal{L}/\mathfrak{a}_q(0) = \kappa$. *The zeros of* $\mathfrak{a}_q(0)$ *in* $\bar{X}$ *satisfy the equations in* $\bar{X}$

$$Q^{-\chi} b_\chi^-(t) = 0, \qquad \chi \in X, \tag{2.10}$$

such that $(\eta, \chi) > 0$ *and vice versa. Hence they coincide with the set of critical points* $Cr(L_\eta)$ *for the function* $\Phi(t)$ *(see (1.5)). The number of such points is equal to* κ.

The subspace $\mathfrak{a}_q(u)$ can be regarded as a $\boldsymbol{C}[u^\chi|_{\chi \in X, (\eta, \chi) > 0}]$-module in $\boldsymbol{C}[u^\chi|_{\chi \in X, (\eta, \chi) > 0}] \otimes \mathcal{L}$. Then $\mathcal{L}/\mathfrak{a}_q(u)$ being a perturbation of $\mathcal{L}/\mathfrak{a}_q(0)$ from $u=0$ to $u \neq 0$, the inequality for the semi-continuity of dimension holds under the finiteness condition. Indeed, let $\mathfrak{h}$ be the linear subspace of $\mathcal{L}$ spanned by $\varphi_1, \cdots, \varphi_\kappa$ as in Lemma 1.2. Similarly like Lemma 2.1 we have

Lemma 2.4. *There exists a Laurent polynomial* $G_0(u|a, a')$ *in* u_j, a_j *and* a'_j *which is a resultant of* $\mathfrak{a}_q(u)$ *with respect to the basis* $\mathfrak{h}$ *such that* $G_0(0|a, a')$ *is not identically zero and that*

$$G_0(u|a, a') t_r^{\pm 1} \varphi_j(t) \equiv 0 \bmod (\mathfrak{h} + \mathfrak{a}_q(u)). \tag{2.11}$$

for all $1 \leq r \leq n$, $1 \leq j \leq \kappa$.

Hence under the additional assumption

Assumption 4. $G_0(uq^\chi|a, a') \neq 0$ *for all* $\chi \in X$,

Lemma 2.5. $\dim \mathcal{L}/\mathfrak{a}_q(u) \leq \kappa$.

From Corollary to Lemma 2.2

$$\dim H^n(\Omega^\cdot, \nabla) \leq \kappa. \tag{2.12}$$

From (1.8) and from (2.12)

Theorem. *Under Assumptions* 1–4, *we have* $\dim H^n(\Omega^\cdot, \nabla) = \kappa$ *and*

$$H^n(\Omega^\cdot, \nabla) \simeq \mathcal{L}/\mathfrak{a}_q(u). \tag{2.13}$$

In our proof of Lemma 2.4, the notions of *Newton polyhedra and Minkowski sum of convex polytopes* are essential. In fact, there exists a finite rational convex polyhedron K in $\check{X}_{\boldsymbol{R}}$ bounded by the hyperplanes $(\eta, \chi) \leq C_\chi$, $C_\chi \in \boldsymbol{R}$ for $\chi \in Y$ and satisfying the following: i) $\Delta(\varphi_j)$, $\Delta(b_\chi^\pm) \subset K$, where $\Delta(\varphi)$ denotes the Newton polyhedron of $\varphi \in \mathcal{L}$. ii) Let S_χ be the convex hull of the set of points $\lambda \in \check{X}_{\boldsymbol{R}}$ such that $\lambda + \Delta(b_\chi^-) \subset K$. We denote by $\boldsymbol{C}\langle\Omega\rangle$ the linear space spanned by $t^\eta, \eta \in \Omega \cap \check{X}$ for a subset Ω in $\check{X}_{\boldsymbol{R}}$. Then the map ι from $\sum_{\substack{\chi \in Y \\ (\eta, \chi) > 0}} \boldsymbol{C}\langle S_\chi \rangle + \mathfrak{h}$ to $\boldsymbol{C}\langle K \rangle$:

$$\{(\psi_\chi)_{\substack{\chi \in Y \\ (\eta, \chi) > 0}}, (c_j)_{1 \leq j \leq \kappa}\} \longrightarrow \sum_{j=1}^\kappa c_j \varphi_j + \sum_{\substack{\chi \in Y \\ (\eta, \chi) > 0}} \{(Q^{-\chi} b_\chi^-) - u^\chi b_\chi^+ Q^\chi\} \psi_\chi \tag{2.14}$$

is surjective, where $c_j \in \boldsymbol{C}$.

Lemma 2.1 can be proved similarly.

Remark. When q tends to 1, then $\mathfrak{a}_1(u; L, L')$ and $\mathfrak{a}_1(u)$ become ideals in $\mathcal{L}$. However $\kappa(=\dim \mathcal{L}/\mathfrak{a}_1(u))$ does not equal $n!$ times the Minkowski mixed volume v_n of the Newton polyhedra $\Delta(b_{\chi_j}^-)$, $1 \leq j \leq n$. We cannot apply Bernshtein's theorem (see [4]) to our case since $\mathfrak{a}_1(u)$ and $\mathfrak{a}_1(u; L, L')$ are degenerate. $\dim \mathcal{L}/\mathfrak{a}_1(u)$ is generally smaller than $n!\, v_n$. The latter depends on the choice of the basis $\chi_1, \cdots, \chi_n$. It seems interesting to give a geometric meaning to κ.

3. Example.

(i) $\Phi = \Pi_{j=1}^n t_j^{\alpha_j} \Pi_{0 \leq i < j \leq n} \dfrac{(a'_{i,j} t_j / t_i)_\infty}{(a_{i,j} t_j / t_i)_\infty}$, for $t_0 = 1$ and $m = \binom{n+1}{2}$. $\mu_j(\chi) = \nu_k - \nu_l$ for $k \neq l$ (we put $\nu_0 = 0$). $[\mu_{j_1}, \cdots, \mu_{j_n}] = \pm 1$, or 0. $\sum_{j=1}^m \mu_j(\chi_r) \mu_j(\chi_s)$ equal n or -1 according as $r = s$ or $r \neq s$. κ is then equal to $(n+1)^{n-1}$. This case has been investigated in more detail in [2].

(ii) $\Phi = \Pi_{j=1}^n t_j^{\alpha_j} \dfrac{(a'_{0,j} t_j)_\infty}{(a_{0,j} t_j)_\infty} \Pi_{1 \leq i < j \leq n} \dfrac{(a'_{i,j} t_j / t_i)_\infty (b'_{i,j} t_i t_j)_\infty}{(a_{i,j} t_j / t_i)_\infty (b_{i,j} t_i t_j)_\infty}$. $m = n^2$ and $\sum_{j=1}^m \mu_j(\chi_r) \mu_j(\chi_s) = (2n-1)\delta_{r,s}$. Hence $\kappa = (2n-1)^n$. This case satisfies Assumptions 1–4 which implies $\dim H^n(\Omega^\cdot, \nabla) = \kappa$.

It seems interesting to evaluate the resultants $G_0(u|a, a')$ for these two cases.

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