56. On the Number of Apparent Singularities

By Akihiro SAEKI
Department of Mathematics, Faculty of Science, University of Tokyo

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§0. **Introduction.** Let M be a closed Riemann surface of genus g, S a finite subset of M and ρ a representation of the fundamental group $\pi_1(M-S)$ on C^n. The Riemann-Hilbert problem states as follows:

Find a Fuchsian linear differential equation of order n with singularities on S and having ρ as its monodromy representation.

When S consists of m distinct points, the set of Fuchsian linear differential equations of order n having their singuralities only on S has a structure of a complex manifold of dimension $2^{-1}n(n+1)m+n^2(g-1)$ (Kita [3]), whereas the totality of equivalence classes of complex n-dimensional irreducible representations of the fundamental group $\pi_1(M-S)$ forms a complex manifold of dimension $n^2(m+2g-2)+1$. So, in general, we must introduce apparent singulalities to solve the Riemann-Hilbert problem. On the number of apparent singulalities, the following result is known ([4]):

Theorem 1 (Ohtsuki). *If the representation ρ is irreducible and if the local representation at a point of S induced by ρ is semi-simple, then there exists a Fuchsian linear equation of order n on M which has the given representation ρ as its monodromy representation and has at most*

$$2^{-1}n(n-1)m+n^2(g-1)+1$$

apparent singularities.

In this paper, without assuming that the local representation at a point of S induced by ρ is semi-simple, we show the following theorem:

Main theorem. *Let M, S and ρ be as above Let d be the greatest common divisor of complex dimensions of all invariant subspaces of the local representation induced by ρ at each point of S. Then there exists a Fuchsian linear equation of order n on M which has the given representation ρ as its monodromy representation and has at most*

$$2^{-1}n(n-1)m+n^2(g-1)+d$$

apparent singularities.

Since d is at most n, we have easily the

Corollary. *There exists a Fuchsian linear equation of order n on M which has the given representation ρ as its monodromy representation and has at most*

$$2^{-1}n(n-1)m+n^2(g-1)+n$$

apparent singularities.

§1. **The method of Deligne.** To establish the main theorem, we recall the proof of Theorem 1 (Ohtsuki [4]).

Deligne's method. 1) The representation ρ defines a local system V' of n-dimensional complex vector spaces over $M-S$.

2) V' determines canonically a holomorphic vector bundle $\boldsymbol{V}'$ over $M-S$ with a holomorphic connection ∇' such that

$$V'=\{\xi \in \boldsymbol{V}' \mid \nabla'\xi=0\}.$$

3) Then the pair $(\boldsymbol{V}', \nabla')$ extends to a pair $(\boldsymbol{V}, \nabla)$, where $\boldsymbol{V}$ is a holomorphic vector bundle over M and ∇ is a meromorphic connection on $\boldsymbol{V}$ such that the restriction $(\boldsymbol{V}|_{M-S}, \nabla|_{\boldsymbol{V}|M-S})$ is equivalent to $(\boldsymbol{V}', \nabla')$.

4) To obtain an ordinary linear differential equation, we take a holomorphic section ψ of the dual bundle $\boldsymbol{V}^*$ such that $\psi(V')$ is isomorphic to V' as local systems, where we consider $\psi(V')$ as a subsheaf of $\mathcal{O}_{M-S}$. Then a differential equation having $\psi(V')$ as the solution sheaf gives a solution of the Riemann-Hilbert problem.

$(\boldsymbol{V}, \nabla)$ is not uniquely determined by $(\boldsymbol{V}', \nabla')$. We use this fact in the proof of the main theorem.

§2. Differential equation with the given solution sheaf. Let $(\boldsymbol{V}, \nabla)$ be an extension of the pair $(\boldsymbol{V}', \nabla')$ with a non-zero holomorphic section $\psi \in \Gamma(M, \mathcal{O}(\boldsymbol{V}^*))$ of the dual bundle $\boldsymbol{V}^*$ of $\boldsymbol{V}$. It follows from the irreducibility of the given representation ρ that the local system $\psi(V')$ is isomorphic to V'. A differential equation having $\psi(V')$ as the solution sheaf is constructed as follows.

Let $\xi_1, \cdots, \xi_n$ be a local $\boldsymbol{C}$-basis of $V'|_U$ on a local chart (U, z). Then $\langle\psi, \xi_1\rangle, \cdots, \langle\psi, \xi_n\rangle$ form a $\boldsymbol{C}$-basis of $\psi(V')|_U$. The differential equation on (U, z) with the solution sheaf $\psi(V')$ is

$$\det\begin{bmatrix} \langle\psi, \xi_1\rangle & \cdots\langle\psi, \xi_n\rangle & y \\ D\langle\psi, \xi_1\rangle & \cdots D\langle\psi, \xi_n\rangle & Dy \\ & \cdots\cdots\cdots\cdots & \\ D^n\langle\psi, \xi_1\rangle & \cdots D^n\langle\psi, \xi_n\rangle & D^n y \end{bmatrix}=0,$$

where $D=d/dz$. On the other hand, the connection ∇ on the bundle $\boldsymbol{V}$ defines the dual connection on the dual bundle $\boldsymbol{V}^*$ of $\boldsymbol{V}$, which we denote also by ∇. The covariant derivative $\nabla_D\psi$ of ψ by D with respect to the connection ∇ is a local meromorphic section of $\boldsymbol{V}^*$ on U. We can define $(\nabla_D)^k\psi \in \Gamma(U, \mathcal{M}(\boldsymbol{V}^*))$ by

$$(\nabla_D)^k\psi=\nabla_D((\nabla_D)^{k-1}\psi).$$

We have

$$D^k\langle\psi, \xi_j\rangle=\langle(\nabla_D)^k\psi, \xi_j\rangle \qquad (j=1, \cdots, n),$$

since, for any meromorphic section $\zeta \in \Gamma(U, \mathcal{M}(\boldsymbol{V}^*))$,

$$D\langle\zeta, \xi_j\rangle=\langle\nabla_D\zeta, \xi_j\rangle+\langle\zeta, \nabla_D\xi_j\rangle \quad \text{and} \quad \nabla_D\xi_j=0.$$

Thus we obtain the differential equation:

$$\det\begin{bmatrix} \langle\psi, \xi_1\rangle & \cdots\langle\psi, \xi_n\rangle & y \\ \langle\nabla_D\psi, \xi_1\rangle & \cdots\langle\nabla_D\psi, \xi_n\rangle & Dy \\ & \cdots\cdots\cdots\cdots & \\ \langle(\nabla_D)^n\psi, \xi_1\rangle & \cdots\langle(\nabla_D)^n\psi, \xi_n\rangle & D^n y \end{bmatrix}=0. \tag{2.1}$$

§ 3. The Wronskian and the number of apparent singularities. We define the Wronskian $W(\psi, \nabla)$ as follows:

First, we define $W(\psi, \nabla_D)$ by

$$W(\psi, \nabla_D)=\psi\wedge\nabla_D\psi\wedge\cdots\wedge(\nabla_D)^{n-1}\psi.$$

For another local coordinate (U', z'), we have on $U\cap U'\neq\phi$:

$$W(\psi, \nabla_D)=K^{n(n-1)/2}W(\psi, \nabla_{D'}),$$

where $D'=d/dz'$ and $K=dz'/dz$. It follows that a global meromorphic section $W(\psi, \nabla)$ of the line bundle $\det(V^*)\otimes\Omega^{n(n-1)/2}$ is well-defined, Ω being the canonical line bundle of the Riemann surface M:

$$W(\psi, \nabla)\in\Gamma(M, \mathcal{M}(\det(V^*)\otimes\Omega^{n(n-1)/2})).$$

We have from (2.1)

$$\alpha_0 D^n y+\alpha_1 D^{n-1}y+\cdots+\alpha_n y=0,$$

where

$$\alpha_0=\langle W(\psi, \nabla), \xi_1\wedge\cdots\wedge\xi_n\rangle$$
$$\alpha_1=-\langle\psi\wedge\nabla_D\psi\wedge\cdots\wedge(\nabla_D)^{n-2}\psi\wedge(\nabla_D)^n\psi, \xi_1\wedge\cdots\wedge\xi_n\rangle$$
$$\cdots\cdots\cdots\cdots\cdots\cdots.$$

Since $\xi_1\wedge\cdots\wedge\xi_n\neq 0$, this differential equation has singularities only at the zeros and poles of the Wronskian $W=W(\psi, \nabla)$. We see ([4]) that W is holomorphic on $M-S$ and has a pole of order at most $n(n-1)/2$ at $a\in S$. The zeros of W are apparent singularities of the differential equation. We have

$$\begin{aligned}&\#(\text{zeros of } W)-\#(\text{poles of } W)\\ &\quad=c_1(\det(V^*)\otimes\Omega^{n(n-1)/2})\\ &\quad=c_1(V^*)+2^{-1}n(n-1)\,c_1(\Omega)\\ &\quad=c_1(V^*)+2^{-1}n(n-1)(2g-2),\end{aligned}$$

where we consider the first Chern classes as integers:

$$c_1(\det(V^*)\otimes\Omega^{n(n-1)/2}),\quad c_1(V^*),\quad c_1(\Omega)\in H^2(M, Z)\simeq Z.$$

It follows that

$$\begin{aligned}&\#(\text{apparent singularities})\\ &\quad\leqq c_1(V^*)+2^{-1}n(n-1)(m+2g-2).\end{aligned}$$

Thus we arrive at the

Lemma 3.1. *Let (V, ∇) be an extension of (V', ∇') such that the dual bundle V^* of V has a non-zero global holomorphic section $\psi\in\Gamma(M, \mathcal{O}(V^*))$. Then there exists a solution of the Riemann-Hilbert problem which has at most*

$$c_1(V^*)+2^{-1}n(n-1)(m+2g-2)$$

apparent singularities.

§ 4. Proof of the theorem. We estimate the $c_1(V^*)$, show the existence of (V, ∇) with a non-zero global section $\psi\in\Gamma(M, \mathcal{O}(V^*))$ and then complete the proof. We construct an extension (V, ∇) of (V', ∇') in the following manner. Let (U, z) be a simply connected coordinate neighbourhood of $a\in S$ such that $z(a)=0$ and $U\cap S=\{a\}$. Let γ be a generator of the fundamental group $\pi_1(U-a)$ and $A_a\in GL(n, C)$ the image of γ by the local representation induced by ρ at a. Let B_a be a matrix such that

$$A_a = \exp(-2\pi i B_a). \tag{4.1}$$

We define a meromorphic connection ∇^U on the trivial bundle $U \times \boldsymbol{C}^n$ by

$$\nabla^U = d + z^{-1} B_a dz.$$

By (4.1), we can patch (V', ∇') and $(U \times \boldsymbol{C}^n, \nabla^U)$ together. In this way, we obtain an extension (V, ∇) of (V', ∇') to M. By a straight computation, we have an important lemma.

Lemma 4.2. *The first Chern class* $\mathrm{c}_1(V)$ $(\in \mathrm{H}^2(M, \boldsymbol{Z}) \simeq \boldsymbol{Z})$ *is given by*

$$\mathrm{c}_1(V) = -\Sigma_{a \in S} \operatorname{tr}(B_a).$$

If the Jordan normal form J_a of A_a is of the following form:

$$J_a = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_\mu \end{bmatrix} \qquad A_k = \begin{bmatrix} \alpha_k & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \alpha_k \end{bmatrix} \in GL(\delta_k, \boldsymbol{C})$$

$$\delta_1 + \cdots + \delta_\mu = n$$

then the Jordan normal form L_a of B_a can be written as:

$$L_a = \begin{bmatrix} B_1 & & & 0 \\ & B_2 & & \\ & & \ddots & \\ 0 & & & B_\mu \end{bmatrix} \qquad B_k = \begin{bmatrix} \beta_k & & & * \\ & \ddots & & \\ & & \ddots & \\ 0 & & & \beta_k \end{bmatrix}.$$

Note that for each $\beta_k \in \boldsymbol{C}$,

$$\alpha_k = \exp(-2\pi i \beta_k).$$

We can choose for each β_k an arbitray branch of the logarithm. If B'_a is another matrix such that

$$A_a = \exp(-2\pi i B'_a), \tag{4.3}$$

then for some $\nu \in \boldsymbol{Z}$,

$$\operatorname{tr}(B'_a) - \operatorname{tr}(B_a) = \nu d_a, \tag{4.4}$$

where d_a is the greatest common divisor of $\delta_1, \cdots, \delta_\mu$. Conversely, for any $\nu \in \boldsymbol{Z}$, there exists a matrix B'_a satisfying (4.3) and (4.4). From (4.2), we have the following lemma.

Lemma 4.5. *For some* $c \in \boldsymbol{Z}$, $\mathrm{c}_1(V)$ *can take any value in*

$$\{c + \nu d \mid \nu \in \boldsymbol{Z}\},$$

where d is the greatest common divisor of the set $\{d_a \mid a \in S\}$.

On the other hand, from the Riemann-Roch theorem, we deduce the following:

Lemma 4.6. *If*

$$\mathrm{c}_1(V^*) \geqq n(g-1) + 1,$$

then there exists a non-zero section $\psi \in \Gamma(M, \mathcal{O}(V^*))$.

By (4.5), we can choose a pair (V, ∇) such that

$$n(g-1) + 1 \leqq \mathrm{c}_1(V^*) \leqq n(g-1) + d.$$

By (3.1), (4.6) and this fact, we complete the proof. If the local representation at some point of S induced by ρ is semi-simple, then $d=1$ and we have Theorem 1.

References

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