# 31. A Two-Parameter Quantization of GL(n)
## (Summary)

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1. One-parameter quantizations (or $q$-analogues) of the general linear group $GL(n)$ are known in two ways. The standard one arises as a dual Hopf algebra to the Drinfeld-Jimbo quantized enveloping algebra $U_q(\mathfrak{gl}(n))$ and is studied by many authors [5] [2] [6] [9] [8] [3]. The second one was defined by Dipper-Donkin [1]. One can define the quantum determinant for both quantizations. It is central in the first case, but not in the latter case.

We construct a two-parameter quantization $GL_{\alpha,\beta}(n)$ of $GL(n)$ depending on two units $\alpha$, $\beta$ in the base ring. The above known $q$-analogues are obtained as special cases by taking $(q, q)$ and $(1, q)$ as $(\alpha, \beta)$ respectively. Further, we construct a two-parameter quantized enveloping algebra $U_{\alpha,\beta}$ associated with $GL_{\alpha,\beta}(n)$. The Drinfeld-Jimbo algebra $U_q(\mathfrak{gl}(n))$ is obtained as a quotient Hopf algebra of $U_{q,q}$.

2. We work over a commutative ring $k$. Let $\alpha$ and $\beta$ be two units in $k$. Let $M_{\alpha,\beta}$ be the $k$-algebra defined by $n^2$ generators $x_{ij}$ $(1\leq i, j\leq n)$ and the following relations:

(2.1) $x_{ik}x_{ij}=\alpha x_{ij}x_{ik}$ if $j<k$.

(2.2) $x_{jk}x_{ik}=\beta x_{ik}x_{jk}$ if $i<j$.

(2.3) $x_{jk}x_{il}=\beta\alpha^{-1}x_{il}x_{jk}$, $x_{jl}x_{ik}-x_{ik}x_{jl}=(\beta-\alpha^{-1})x_{il}x_{jk}$ if $i<j$ and $k<l$.

The algebra $M_{\alpha,\beta}$ is a (non-commutative) polynomial algebra in $x_{ij}$ in any ordering. This means if $w_1, \cdots, w_N$ $(N=n^2)$ is an arbitrary arrangement of $x_{ij}$ $(1\leq i, j\leq n)$, then the monomials $w_1^{e_1}\cdots w_N^{e_N}$ $(e_i\in \mathbf{N})$ form a free $k$-base for $M_{\alpha,\beta}$. If $k$ is an integral domain, there is no non-zero divisor in $M_{\alpha,\beta}$.

The algebra $M_{\alpha,\beta}$ has a bialgebra structure such that

$$\Delta x_{ij}=\sum_{s=1}^{n} x_{is}\otimes x_{sj}, \quad \varepsilon x_{ij}=\delta_{ij}.$$

The *quantum determinant* $g=|X|$ is defined by

$$g=\sum_{\sigma}(-\beta)^{-l(\sigma)}x_{\sigma(1),1}\cdots x_{\sigma(n),n}$$
$$=\sum_{\sigma}(-\alpha)^{-l(\sigma)}x_{1,\sigma(1)}\cdots x_{n,\sigma(n)}$$

where $\sigma$ ranges over all permutations of $n$ letters, and $l(\sigma)$ denotes the number of inversions. It is a group-like element, i.e., we have

$$\Delta g=g\otimes g, \quad \varepsilon g=1.$$

It is a non-zero divisor of $M_{\alpha,\beta}$ and we have

$$x_{ij}g=(\beta\alpha^{-1})^{i-j}gx_{ij}.$$

Hence the powers of $g$ satisfy the left and right Ore condition. The localization

$$A_{\alpha,\beta}=M_{\alpha,\beta}[g^{-1}]$$

is a Hopf algebra containing $M_{\alpha,\beta}$ as a subbialgebra. The antipode $S$ is defined by

$$S(x_{ij})=(-\beta)^{j-i}g^{-1}|X_{ji}|=(-\alpha)^{j-i}|X_{ji}|g^{-1}$$

where $|X_{ji}|$ denotes the quantum determinant of the $(n-1)\times(n-1)$ minor obtained by removing the $j$-th row and the $i$-th column. We have

$$S^2(x_{ij})=(\alpha\beta)^{j-i}x_{ij}.$$

Let $GL_{\alpha,\beta}(n)$ be the quantum group over $k$ represented by the Hopf algebra $A_{\alpha,\beta}$. The standard (resp. the Dipper-Donkin) quantum $GL(n)$ is obtained as a special case if we take $(\alpha,\beta)=(q,q)$ (resp. $(1,q)$).

3. From now on, assume $\alpha\beta-1$ is a unit, too. Let $U_{\alpha,\beta}$ be the $k$-algebra defined by generators $a_i$, $a_i^{-1}$, $b_i$, $b_i^{-1}$ $(1\leq i\leq n)$, $e_j$, $f_j$ $(1\leq j<n)$ and the following relations:

(3.1) $a_i$, $b_i$ $(1\leq i\leq n)$ commute with one another and

$$a_ia_i^{-1}=a_i^{-1}a_i=b_ib_i^{-1}=b_i^{-1}b_i=1.$$

(3.2) $a_ie_j=\alpha^{\delta_{ij}-\delta_{i,j+1}}e_ja_i,\quad b_ie_j=\beta^{\delta_{ij}-\delta_{i,j+1}}e_jb_i,$

$a_if_j=\alpha^{-\delta_{ij}+\delta_{i,j+1}}f_ja_i,\quad b_if_j=\beta^{-\delta_{ij}+\delta_{i,j+1}}f_jb_i.$

(3.3) $[e_j,f_k]=\dfrac{\delta_{jk}}{\alpha-\beta^{-1}}(a_kb_{k+1}^{-1}-a_{k+1}b_k^{-1}).$

(3.4) $[e_j,e_k]=[f_j,f_k]=0\qquad \text{if } |j-k|>1.$

(3.5) $[[e_j,e_{j+1}]_\alpha,e_j]_\beta=[[e_{j+1},e_j]_\beta,e_{j+1}]_\alpha=0,$

$[[f_j,f_{j+1}]_\beta,f_j]_\alpha=[[f_{j+1},f_j]_\alpha,f_{j+1}]_\beta=0\quad (1\leq j\leq n-2).$

In (3.5), we mean

$$[x,y]_\alpha=xy-\alpha yx,\quad [x,y]_\beta=xy-\beta yx.$$

The algebra $U_{\alpha,\beta}$ has a bialgebra structure such that

$$\Delta e_j=1\otimes e_j+e_j\otimes a_jb_{j+1}^{-1},$$
$$\Delta f_j=f_j\otimes 1+a_{j+1}b_j^{-1}\otimes f_j\quad (1\leq j<n)$$

and $a_i$, $b_i$ $(1\leq i\leq n)$ are group-like. It is a Hopf algebra.

There is a canonical triangular decomposition

$$U_{\alpha,\beta}=U_{\alpha,\beta}^-\otimes U_{\alpha,\beta}^0\otimes U_{\alpha,\beta}^+$$

similarly as the Drinfeld-Jimbo algebra. The $\pm$ parts admit free $k$-bases similar to the one described in [10] if $\alpha\beta+1$ is invertible in addition. Lusztig's representation theory in [4] can be generalized to $U_{\alpha,\beta}$.

If $a=b=q$, then $a_ib_i^{-1}$ $(1\leq i\leq n)$ are central group-like elements. The quotient Hopf algebra of $U_{q,q}$ by the Hopf ideal $(a_ib_i^{-1}-1,\ 1\leq i\leq n)$ is identified with the Drinfeld-Jimbo Hopf algebra $U_q(\mathfrak{gl}(n))$.

Similarly, if $\alpha=1$ and $\beta=q$, then $a_i$ $(1\leq i\leq n)$ are central group-like, and one can construct the quotient Hopf algebra of $U_{1,q}$ by the Hopf ideal $(a_i-1,\ 1\leq i\leq n)$. This quotient Hopf algebra is associated with the Dipper-Donkin quantum $GL(n)$ (see 4).

There is also a two-parameter analogue of $U_q(\mathfrak{sl}(n))$. Let $U'_{\alpha,\beta}$ be the subalgebra of $U_{\alpha,\beta}$ generated by $e_j$, $f_j$, $a_jb_{j+1}^{-1}$, $a_j^{-1}b_{j+1}$, $a_{j+1}b_j^{-1}$, $a_{j+1}^{-1}b_j$ $(1\leq j<n)$.

It is a Hopf subalgebra stable under the adjoint action

$$Ad(h)=\sum h_{(1)}(-)S(h_{(2)}), \quad h\in U_{\alpha,\beta}.$$

Hence one can construct the quotient Hopf algebra $U_{\alpha,\beta}/U'_{\alpha,\beta}$. It is isomorphic to the group (Hopf) algebra of $\mathbf{Z}^2$. If $\alpha=\beta=q$, the image of $U'_{q,q}$ in $U_q(\mathfrak{gl}(n))$ is precisely $U_q(\mathfrak{sl}(n))$. If $\alpha=1$ and $\beta=q$, $U'_{1,q}$ maps surjectively to the quotient Hopf algebra by the ideal $(a_i-1,\ 1\leq i\leq n)$.

**4.** The Hopf algebra $U_{\alpha,\beta}$ is associated with the quantum group $GL_{\alpha,\beta}(n)$ in the following sense. There is a Hopf pairing (see [9])

$$\langle -,-\rangle :\ U_{\alpha,\beta}\times A_{\alpha,\beta}\to k$$

such that we have

$$\langle a_i, x_{st}\rangle=\delta_{st}\alpha^{\delta_{is}}, \quad \langle b_i, x_{st}\rangle=\delta_{st}\beta^{\delta_{is}},$$
$$\langle e_j, x_{st}\rangle=\delta_{js}\delta_{j+1,t}, \quad \langle f_j, x_{st}\rangle=\delta_{j+1,s}\delta_{jt}.$$

If $k$ is a field, this pairing induces Hopf algebra maps

$$U_{\alpha,\beta}\to A^{\circ}_{\alpha,\beta}, \quad A_{\alpha,\beta}\to U^{\circ}_{\alpha,\beta}$$

adjoint with each other. Here $(\ )^{\circ}$ denotes the dual Hopf algebra in the sense of Sweedler [7].

In the special cases $(\alpha,\beta)=(q,q)$ or $(1,q)$, one can replace $U_{q,q}$ or $U_{1,q}$ by the previously mentioned quotient Hopf algebras.

## References


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