29. An Additive Theory of the Zeros of the Riemann Zeta Function

By Akio FUJII
Department of Mathematics, Rikkyo University

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The purpose of the present article is to present an additive theory of the zeros of the Riemann zeta function $\zeta(s)$. The details with some more general results will appear elsewhere.

We recall first the well-known Riemann-von Mangoldt formula for the number $N(T)$ of the zeros of $\zeta(s)$ in $0<\mathrm{Re}\, s<1$, $0<\mathrm{Im}\, s\leq T$ (cf. p. 179 and p. 256 of Titchmarsh [8]).

$$\text{(A)}: \quad N(T)=\frac{1}{2\pi}T\log T-\frac{1+\log 2\pi}{2\pi}T+\frac{7}{8}+O\left(\frac{1}{T}\right)+S(T),$$

where $T>T_0$ and $S(T)=(1/\pi)\arg\zeta((1/2)+iT)=O(\log T)$.

Under the Riemann Hypothesis (R.H.), it is well-known that $S(T)=O(\log T/\log\log T)$.

We recall second Landau's theorem on an arithmetic connection of the zeros with a prime number (cf. Landau [7]).

$$\text{(B)}: \quad \sum_{0<\gamma\leq T} x^{\rho}=-\frac{T}{2\pi}\Lambda(x)+O(\log T)$$

for any $x>1$, where $\rho=\beta+i\gamma$ denotes a zero of $\zeta(s)$ and $\Lambda(x)=\log p$, if $x=p^k$, with a prime number p and a positive integer k, and $=0$ otherwise.

Under R.H., this can be improved as follows (cf. Fujii [2] and [6]).

(B′) (Under R.H.): For any $x>1$ and $T>T_0$,

$$\sum_{0<\gamma\leq T} x^{(1/2)+i\gamma}=-\frac{T}{2\pi}\Lambda(x)+\frac{x^{(1/2)+iT}\log(T/2\pi)}{2\pi i\log x}+O\left(\frac{\log T}{\log\log T}\right).$$

We recall next the following result on an arithmetic connection of the zeros with a rational number (cf. Fujii [1], [2], [3] and [4]). We put $e(x)=e^{2\pi i x}$.

(C) (Under R.H.): Let K be an integer ≥ 1. Then we have

$$\lim_{T\to\infty}\frac{1}{(T/2\pi)^{(1/2)(1+(1/K))}}\sum_{0<\gamma\leq T} e\left(\frac{\gamma}{2\pi K}\log\frac{\gamma}{2\pi e\alpha K}\right)$$

$$=\begin{cases} -e^{\pi i/4}C\left(\dfrac{a}{q}, K\right) & \text{if } \alpha=\dfrac{a}{q} \text{ with integers } a \text{ and } q\geq 1,\ (a,q)=1 \\ 0 & \text{if } \alpha \text{ is irrational } (>0), \end{cases}$$

where we put

$$C\left(\frac{a}{q}, K\right)=2\cdot K^{(1/2)(1-(1/K))}\overline{S\left(\frac{a}{q}, K\right)}(K+1)^{-1}\varphi(q)^{-1}\left(\frac{a}{q}\right)^{-1/(2K)}$$

and

$$S\left(\frac{a}{q}, K\right)=\sum_{\substack{b=1\\(b,q)=1}}^{q} e\left(\frac{a}{q}b^K\right)$$

and $\varphi(q)$ is the Euler function.

Finally, we recall the following result which shows that the vertical distribution of the zeros of $\zeta(s)$ is deeply connected with the Generalized Riemann Hypothesis (G.R.H.) for the Dirichlet L-functions $L(s,\chi)$ (cf. Fujii [3] and [4]).

(D) (Under R.H.): Let q be an integer ≥ 3. Suppose that K is an integer ≥ 5. Then G.R.H. for all $L(s,\chi^K)$ with a character χ mod q is equivalent to the relation

$$\sum_{0<\gamma\leq T} e\left(\frac{\gamma}{2\pi K}\log\frac{\gamma}{2\pi e\frac{a}{q}K}\right)=-e^{\pi i/4}C\left(\frac{a}{q}, K\right)\left(\frac{T}{2\pi}\right)^{(1/2)(1+(1/K))}+O(T^{(1/2)+\varepsilon})$$

for any positive ε, any integer a with $1\leq a\leq q$ and $(a,q)=1$ and for $T>T_0$.

We are now in a position to state our problem.

Problem. *Extend* (A), (B), (B′), (C) *and* (D) *to the sums of the zeros. In particular, are the properties* (B), (B′), (C) *and* (D) *inherited to the sums of the zeros?*

We denote the positive imaginary parts of the zeros ρ or ρ' of $\zeta(s)$ by γ or γ', respectively. We shall state our results.

Theorem 1.

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ \gamma+\gamma'\leq T}}\cdot 1=\frac{1}{8\pi^2}T^2\log^2 T-\frac{1}{8\pi^2}(3+2\log 2\pi)T^2\log T$$
$$+\frac{1}{16\pi^2}(7+6\log 2\pi+2\log^2 2\pi-2\zeta(2))T^2+O\left(T\frac{\log^2 T}{\log\log T}\right).$$

Theorem 1′ (Under R.H.).

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ \gamma+\gamma'\leq T}}\cdot 1=\frac{1}{8\pi^2}T^2\log^2 T-\frac{1}{8\pi^2}(3+2\log 2\pi)T^2\log T,$$
$$+\frac{1}{16\pi^2}(7+6\log 2\pi+2\log^2 2\pi-2\zeta(2))T^2+O(T\log T).$$

Theorem 1′ implies the following corollary.

Cororally 1. *For any $T>T_0$, there exist γ and γ' such that*

$$|T-(\gamma+\gamma')|<\frac{C}{\log T},$$

where C is some positive constant.

We should recall that Cororally 1 can be proved without using R.H. It is proved in the author's [5], as an application of the author's mean value theorem on

$$\int_0^T (S(t+h)-S(t))^{2k}dt \qquad \text{for } h=\frac{C}{\log T}.$$

We shall next state an analogue of (B). We first notice the simplest analogue of (B).

Theorem 2. *For any $x>1$ and $T>T_0$, we have*

$$\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma+\gamma'\le T}} x^{\rho+\rho'}=\frac{1}{8\pi^2}T^2\Lambda^2(x)+O(T\log^2 T).$$

If we assume R.H., then we can refine Theorem 2 as follows.

Theorem 3 (Under R.H.). *Suppose that* $T>T_0$, $x>1$ *and* $(1/\log x)\ll T$. *Then we have*

$$\begin{aligned}\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma+\gamma'\le T}} & x^{i(\gamma+\gamma')}\\ =&\frac{1}{8\pi^2}\frac{\Lambda^2(x)}{x}T^2+\frac{x^{iT}}{4\pi^2 i\log x}T\log^2 T\\ &+O\left(\frac{T\log^2 T}{\operatorname{loglog} T}(\log(3x)+x^{1/\operatorname{loglog} T})\right)+O(T\log T(\sqrt{x}\log(3x)+B(x,T)))\\ &+O\left(B(x,T)\left(\sqrt{x}\log(3x)+\sqrt{x}\sqrt{\frac{\log T}{\operatorname{loglog} T}}+x^{1/\operatorname{loglog} T}\log(3x)\frac{\log T}{\operatorname{loglog} T}\right)\right)\\ &+O\left(x\log^2(3x)+x\frac{\log T}{\operatorname{loglog} T}\right)+O\left(\frac{T\log T}{\log x}\operatorname{Min}\left(\frac{1}{\log x},\log T\right)\right),\end{aligned}$$

where we put

$$B(x,T)=\frac{1}{\sqrt{x}}\sum_{\substack{(x/2)<n<2x\\ n\ne x}}\Lambda(n)\operatorname{Min}\left(T,\frac{1}{\left|\log\dfrac{x}{n}\right|}\right).$$

This is an analogue of (B′) stated above.

Finally, we notice the following theorem which involves the information on both the analogue of (C) and that of (D).

Theorem 4 (Under R.H.). *Let* b *be a positive number* ≤ 2 *and* $B=1/b$. *Let* α *be a positive number satisfying* $T^{1-(4/b)}\ll\alpha\ll T^{4/5}$. *Then*

$$\begin{aligned}\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma+\gamma'\le T}} & e\left(\frac{b}{2\pi}(\gamma+\gamma')\log\frac{\gamma+\gamma'}{2\pi e\alpha}\right)\\ =&\frac{\sqrt{2}\,\alpha^{3/2}}{(1-i)\sqrt{b}}\sum_{k<(T/2\pi\alpha)^b}\Lambda^2(k)k^{(3B/2)-1}\cdot e(-b\alpha k^B)\\ &+O\left(T^{1+(2/5)}\left(\frac{T}{\alpha}\right)^{2b/5}\log T\cdot\left(1+\frac{\alpha^{-b}}{T^{1-b}}\operatorname{loglog} T\right)\right)\\ &+O(\alpha^{-b/2}(1+\alpha^{-b/2}\log T)T^{1+(b/2)}\log^2 T)+O(T^{(3/2)-b}\alpha^b\log^2 T)\\ &+O\left(\alpha^{-b}T^{1+(2/5)}\left(\frac{T}{\alpha}\right)^{-b/10}\log^2 T\right).\end{aligned}$$

From Theorem 4, we get the following corollary.

Cororally 2 (Under R.H.). *Let* α *be a positive number. If* K *is an integer* ≥ 5, *then we have*

$$\begin{aligned}\lim_{T\to\infty} & \frac{1}{(T/2\pi)^{3/2}\log T}\sum_{\substack{0<\gamma,\gamma'\le T\\ \gamma+\gamma'\le T}} e\left(\frac{\gamma+\gamma'}{2\pi K}\log\frac{\gamma+\gamma'}{2\pi e\alpha K}\right)\\ =&\begin{cases}\dfrac{1}{\varphi(q)}\dfrac{\sqrt{2}}{1-i}\dfrac{2}{3K^{3/2}}\displaystyle\sum_{\substack{\chi;q\\ \chi^K=\chi_0}}\chi(a)\bar{\tau}(\chi) & \text{if } \alpha=\dfrac{a}{q},\ (a,q)=1,\ a,q\ge 1\\ 0 & \text{if } \alpha \text{ is irrational,}\end{cases}\end{aligned}$$

where χ *runs over all characters* mod q, χ_0 *is the principal character* mod q

and we put

$$\tau(\chi)=\sum_{\substack{b=1\\(b,q)=1}}^{q}\chi(b)e\left(\frac{b}{q}\right).$$

Finally we shall state our analogue of (D) as follows.

Cororally 3 (Under R.H.). *Let q be an integer ≥ 3 and K be an integer ≥ 9. Then* G.R.H. *for all $L(s,\chi^K)$ with a character χ* mod q *is equivalent to the relation*

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ \gamma+\gamma'\leq T}} e\left(\frac{\gamma+\gamma'}{2\pi K}\log\frac{\gamma+\gamma'}{2\pi e\frac{a}{q}K}\right)$$
$$=\frac{1}{\varphi(q)}\frac{\sqrt{2}}{1-i}\frac{2}{3K^{3/2}}\left(\frac{T}{2\pi}\right)^{3/2}\left(\log\frac{T}{2\pi\frac{a}{q}K}-\frac{2}{3}\right)\sum_{\substack{\chi;q\\ \chi^K=\chi_0}}\bar{\tau}(\chi)\chi(a)$$
$$+O(T^{(3/2)-(1/2K)+\varepsilon})$$

for any positive ε, any integer a with $1\leq a\leq q$ and $(a,q)=1$ and for $T>T_0$.

Thus we have seen that the addition does not destroy such arithmetic natures as the distribution of the zeros has originally.

Cororally 2 should hold also for $K=1,2,3$ and 4 and Cororally 3 should hold also for $K=1,2,3,\cdots$ and 8.

Finally, we shall make some concluding remarks.

1. We can extend Theorem 1 to $\sum_{\gamma+\gamma'\leq Y, 0<\gamma,\gamma'\leq T}\cdot 1$ for $T\leq Y\leq 2T$. Similarly, Theorems 2, 3 and 4 and Corollaries 2 and 3 can be extended.

2. We can extend our theorems to more general sums

$$\gamma_1+\gamma_2+\gamma_3+\cdots+\gamma_n,\qquad \text{for } n\geq 2.$$

We can also extend our theorems to the zeros of Dirichlet L-functions.

3. Using Theorems 3 and 4, as in Fujii [3] and [4], we can obtain various mean value theorems like

$$\sum_{\substack{0<\gamma,\gamma'\leq T\\ \gamma+\gamma'\leq T}}\zeta\left(\frac{1}{2}+i(\gamma+\gamma')\right)\quad\text{and}\quad\sum_{\substack{0<\gamma,\gamma'\leq T\\ \gamma+\gamma'\leq T}}\left|L\left(\frac{1}{2}+i(\gamma^*+\gamma'),\chi\right)\right|^2.$$

4. These results can be obtained by extending and using the arguments in [2] and [6].

References

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