91. *Second Microlocal Singularities and Boundary Values of Holomorphic Functions*

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1. Introduction. The theory of the second microlocalization was initiated by M. Kashiwara [9]. He constructed the sheaf of 2-microfunctions and introduced the notion of second singular spectrum for microfunctions (cf. J.M. Bony [1], Kashiwara-Laurent [10]). On the other hand, J. Sjöstrand defined second analytic wave front sets for distributions with FBI transformation (cf. Esser-Laubin [7, 8]).

This paper aims at clarifying the relation between second microlocal singularities of hyperfunctions (or microfunctions) along linear involutive submanifolds of the cotangent bundle and their expressions as boundary values of holomorphic functions.

We introduce a special class of profiles and corresponding tuboids. Then we give a necessary and sufficient condition for a point to be outside of the second analytic wave front set of a hyperfunction. In the subsequent note [12], we will study, with this result, several functorial properties of second analytic wave front sets and will show the two notions of second microlocal singularities are equivalent.

2. Second analytic wave front sets. Let M be an open subset of $\boldsymbol{R}^n_x$ with a complex neighborhood X in $\boldsymbol{C}^n_z$. Then $(x;\xi\cdot dx)$ denotes a point of the cotangent bundle T^*M of M. We identify M with the zero section $T^*_M M$ of T^*M, and set $\dot{T}^*M=T^*M\setminus M$. We define an involutive submanifold V of $\dot{T}^*M$ by

$$V=\{(x;\xi)\in\dot{T}^*M\,;\,\xi_1=\cdots=\xi_d=0\}.$$

To simplify the notation, we put

$$x'=(x_1,\cdots,x_d),\qquad x''=(x_{d+1},\cdots,x_n),$$
$$\xi'=(\xi_1,\cdots,\xi_d),\qquad \xi''=(\xi_{d+1},\cdots,\xi_n),\quad etc.$$

We take coordinates of $T_V T^*M$ as $(x;\xi''\cdot dx'';x'^*\cdot\partial/\partial\xi')$ with $x'^*=(x_1^*,\cdots,x_d^*)\in\boldsymbol{R}^d$.

Definition 2.1. I) Let $u(x)$ be a hyperfunction with compact support. Then we define the FBI (Fourier-Bros-Iagolnitzer) transform and the second FBI transform along V of u by

$$Tu(z,\lambda):=\int u(x)\exp\left\{-\frac{\lambda}{2}(z-x)^2\right\}dx,$$

$$T^2_V u(z,\lambda,\mu):=\int u(x)\exp\left\{-\frac{\lambda\mu}{2}(z'-x')^2-\frac{\lambda}{2}(z''-x'')^2\right\}dx.$$

II) Let $\dot{q}=(\dot{x};\dot{\xi})\in \dot{T}^*M$. Then $\dot{q}\notin \mathrm{WF}_a(u)$ if and only if there exist, $C, r, \varepsilon>0$ for which we have the estimate

$$|Tu(z,\lambda)|\leq C\cdot\exp\left(\frac{\lambda}{2}|\mathrm{Im}\, z|^2-\varepsilon\lambda\right) \qquad (\lambda>0, |z-(\dot{x}-\sqrt{-1}\dot{\xi})|<r).$$

$\mathrm{WF}_a(u)$ is called the *analytic wave front set* of u.

III) Let $\dot{p}=(\dot{x};\dot{\xi}''\cdot dx'';\dot{x}'^*\cdot\partial/\partial\xi')\in \dot{T}_V\dot{T}^*M$. Then $\dot{p}\notin \mathrm{WF}^2_{a,V}(u)$ if there exist positive numbers ε, r, μ_0 and a decreasing function $f:\,]0,\mu_0[\to \boldsymbol{R}$ such that

$$|T^2_V u(z,\lambda,\mu)|\leq \exp\left(\frac{\lambda\mu}{2}|\mathrm{Im}\, z'|^2+\frac{\lambda}{2}|\mathrm{Im}\, z''|^2-\varepsilon\lambda\mu\right)$$

$$(0<\mu<\mu_0,\ \lambda>f(\mu),\ |z'-(\dot{x}'-\sqrt{-1}\,\dot{x}'^*)|<r,\ |z''-(\dot{x}''-\sqrt{-1}\,\dot{\xi}'')|<r).$$

The subset $\mathrm{WF}^2_{a,V}(u)$ is called the *second analytic wave front set* of u along V.

IV) Let $\dot{q}=(\dot{x};\dot{\xi}''\cdot dx'')\in V$. Then $\dot{q}\notin$ 2-singsupp$_V(u)$ if there exists $r>0$ such that for any $\eta>0$ we can find a positive C_η satisfying

$$|Tu(z,\lambda)|\leq C_\eta\cdot\exp\left(\frac{\lambda}{2}|\mathrm{Im}\, z''|^2+\lambda\eta\right) \qquad (\lambda>0, |z-(\dot{x}-\sqrt{-1}(0,\dot{\xi}''))|<r).$$

The set 2-singsupp$_V(u)$ is called the *second analytic singular support* of u along V.

We give several remarks: i) The above definitions are due to J. Sjöstrand (cf. [7, 8], [13]). ii) We can easily justify the above definitions of $\mathrm{WF}_a(\cdot)$, $\mathrm{WF}^2_{a,V}(\cdot)$, 2-singsupp$_V(\cdot)$ for all hyperfunctions without the assumption of compactly supportedness. Moreover we can make sense of $\mathrm{WF}^2_{a,V}(\cdot)$, 2-singsupp$_V(\cdot)$ for microfunctions. iii) Set $\dot{\pi}_V$: $\dot{T}_V\dot{T}^*M\to V$. Then we have for $u\in\mathscr{B}_M$

$$\text{2-singsupp}_V(u)=\dot{\pi}_V(\mathrm{WF}^2_{a,V}(u)).$$

Refer to Esser-Laubin [7, 8] for this. iv) Let $u(x)\in\mathscr{B}_M$. Then $\mathrm{WF}_a(u)\cap V$ enjoys a propagation theorem with respect to x' variables outside of 2-singsupp$_V(u)$. This is implicitly shown in J.M. Bony [3].

3. Second analytic wave front sets and boundary values of holomorphic functions.

3.1. Profiles and tuboids. We follow the notation prepared in §2. But we set $M=\boldsymbol{R}^n_x$, $X=\boldsymbol{C}^n_z$ and $N=\boldsymbol{C}^d_{z'}\times\boldsymbol{R}^{n-d}_{x''}$.

First we briefly recall the classical definitions of profiles and tuboids due to Bros-Iagolnitzer [4, 5, 6].

Definition 3.1. I) A *profile* Λ is an open subset of X with the property:

$$x+\sqrt{-1}y\in\Lambda, \qquad t>0\Longrightarrow x+\sqrt{-1}ty\in\Lambda.$$

We define a projection p: $X\to M$ by $x+\sqrt{-1}y\mapsto x$, and set $\omega=p(\Lambda)$. Then ω is called the *basis* of Λ.

II) A profile Λ is called *convex* if for any $x\in\omega$ the fiber $\Lambda_x:=(p|_\Lambda)^{-1}(x)$ at x is convex.

III) An open subset Ω in X is a *tuboid* with profile Λ if $\Omega\subset\Lambda$ and if for any $K\Subset\Lambda$ there exists $r>0$ satisfying

$$\{x+\sqrt{-1}ty \in X\,;\ 0<t<r,\ x+\sqrt{-1}y \in K\} \subset \Omega.$$

Next we give a special class of profiles and corresponding tuboids.

Definition 3.2. I) An open subset Γ in N is a *semi-profile* with basis ω if

(i) $\omega=\{x \in \boldsymbol{R}^n\,;\ \exists y' \in \boldsymbol{R}^d$ such that $(x'+\sqrt{-1}y', x'') \in \Gamma\}$,

(ii) $\begin{cases}\Gamma_{\dot{x}} :=\{y' \in \boldsymbol{R}^d\,;\ (\dot{x}'+\sqrt{-1}y', \dot{x}'') \in \Gamma\} \\ \text{is an open convex cone in } \boldsymbol{R}^d \text{ for all } \dot{x} \in \omega.\end{cases}$

II) A convex profile Λ with basis ω is called a *generalized V-profile* if there exists a semi-profile Γ satisfying

(iii) $\begin{cases}\text{for any } x \in \omega: (x'+\sqrt{-1}\tilde{y}', x'') \in \Gamma,\ (x'+\sqrt{-1}y', x''+\sqrt{-1}y'') \in \Lambda \\ \qquad\qquad \Longrightarrow (x'+\sqrt{-1}(\tilde{y}'+y'),\ x''+\sqrt{-1}y'') \in \Lambda.\end{cases}$

We set Λ' as the largest semi-profile Γ with basis ω satisfying the condition (iii), and Λ' is refered to as the *V-semi-profile* associated with Λ.

III) Let Γ_1 and Γ_2 be two semi-profiles based on ω. Then $\Gamma_1 \Subset \Gamma_2$ if for any $K \Subset \omega$

$$\{(x'+\sqrt{-1}y', x'') \in \Gamma_1\,;\ x \in K, |y'|=1\} \Subset \Gamma_2.$$

IV) Let Λ be a generalized V-profile. A tuboid Ω with profile Λ is called *strict* with V-profile Λ if for any $K \Subset \Lambda$ and for any semi-profile $\Gamma \Subset \Lambda'$, there exist s_0, $r>0$ such that

$$x+\sqrt{-1}y \in K,\ 0<s<s_0,\ \tilde{y}' \in \Gamma_x, \qquad |\tilde{y}'|<r \Longrightarrow x+\sqrt{-1}(sy+\tilde{y}', sy'') \in \Omega.$$

V) For a conic subset C in $\boldsymbol{R}^l$, we define its *polar set* C^* by

$$C^* :=\{\xi \in \boldsymbol{R}^l \backslash \{0\}\,;\ \xi \cdot y \geq 0 \text{ for any } y \in C\}.$$

3.2. Main theorems. Now we give the two main theorems of this paper.

Theorem 3.3. *Let Ω be a strict tuboid with generalized V-profile Λ, and $f \in \mathcal{O}_X(\Omega)$. Then the boundary value $b(f)$ satisfies*

$$\mathrm{WF}^2_{a,V}(b(f)) \subset \{(x\,;\xi''\,;x'^*) \in \dot{T}_V\dot{T}^*\omega\,;\ (0,\xi'') \in \Lambda^*_x,\ x'^* \in \Lambda'^*_x\}.$$

Theorem 3.4. *Let $u(x) \in \mathcal{B}_M(\omega)$, and $\dot{p}=(\dot{x}\,;\dot{\xi}''\,;\dot{x}'^*) \in \dot{T}_V\dot{T}^*\omega$. If $\dot{p} \notin \mathrm{WF}^2_{a,V}(u)$, there exist generalized V-profiles Λ_j $(j=1,\cdots,N)$ based on ω, strict tuboids Ω_j with Λ_j $(j=1,\cdots,N)$, and $F_j(z) \in \mathcal{O}_X(\Omega_j)$ $(j=1,\cdots,N)$ such that*

$$\dot{x}'^* \notin \Lambda'^*_{j,\dot{x}}, \qquad (\dot{x}\,;(0,\dot{\xi}'')) \notin \mathrm{WF}_a\Big(u-\sum_{j=1}^{N} b(F_j)\Big).$$

Theorem 3.3 is easier to verify. In fact, it suffices to deform the path of integration in the complex domain in calculating the second FBI transform of u. To show Theorem 3.4, we use the following lemma.

Lemma 3.5. *Let $u(x) \in \mathcal{B}_c(M)$, and take a closed convex cone $\Gamma'' \subset \boldsymbol{R}^{n-d}$. Define a holomorphic function $F(z)$ by*

$$F(z)=\frac{1}{(2\pi)^n}\int_{|x'^*|<\delta,\,\xi''\in\Gamma''} \exp\Big(-\frac{|\xi''||x'^*|+|\xi''|}{2}\Big)$$
$$\times Q(\xi'', x'^*, \partial_z)T^2_V u\Big(z'-\frac{\sqrt{-1}x'^*}{|x'^*|},\ z''-\frac{\sqrt{-1}\xi''}{|\xi''|},\ |\xi''|,\ |x'^*|\Big)d\xi'' dx'^*$$

with

$$Q(\xi'', x'^*, \partial_z)=\left(\frac{|\xi''|}{2}-\frac{\sqrt{-1}x'^*\cdot\partial_{z'}}{2|x'^*|^2}\right)\left(\frac{1}{2}-\frac{\sqrt{-1}\xi''\cdot\partial_{z''}}{2|\xi''|^2}\right).$$

Then if $\dot{\xi}''\in \operatorname{Int}\Gamma''$, *we have*

$$(\dot{x}\,;(0,\dot{\xi}''))\notin \mathrm{WF}_a(u-b(F)).$$

In view of this lemma, it suffices to decompose the holomorphic function $F(z)$ into a sum of holomorphic functions with desired properties in Theorem 3.4. Refer to [11] for details.

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