85. A Characterization for Paracompactness

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Introduction. Recently [5, 6] the authors introduced the notion of $B(P, \lambda)$-refinability and used this idea to obtain characterizations for paracompact, subparacompact, metacompact, θ-refinable, collectionwise normal, collectionwise subnormal and strong-collectionwise subnormal spaces. In this paper more general results are obtained in this class of $B(LF, \lambda)$-refinable spaces.

The properties P considered in this paper will be discrete (D) and locally finite (LF). The symbol λ will denote any countable ordinal.

Definition 1. A space X is *$B(P, \lambda)$-refinable* provided every open cover $\mathcal{U}$ of X has a refinement $\mathcal{E}=\cup\{\mathcal{E}_\beta : \beta<\lambda\}$ which satisfies i) $\{\cup\mathcal{E}_\beta : \beta<\lambda\}$ partitions X, ii) for every $\beta<\lambda$, $\mathcal{E}_\beta$ is a relatively P collection of closed subsets of the subspace $X-\cup\{\cup\mathcal{E}_\mu : \mu<\beta\}$, and iii) for every $\beta<\lambda$, $\cup\{\cup\mathcal{E}_\mu : \mu<\beta\}$ is a closed set.

The collection $\mathcal{E}$ is often called a *$B(P, \lambda)$-refinement* of $\mathcal{U}$. Expandable and θ-expandable spaces have been studied in [3, 4, 10, 11].

Definition 2. A space X is *strong-collectionwise subnormal* ($CWSN$) provided every discrete collection $\mathcal{D}$ of closed subsets X has a pairwise disjoint G_δ-expansion which is also a θ-expansion of $\mathcal{D}$.

In [6] the authors have obtained the following.

Theorem 1. *For any strong-CWSN space X, the following are equivalent.*

(a) *X is subparacompact.*

(b) *X is metacompact.*

(c) *X is θ-refinable.*

(d) *X is $B(D, \omega)$-refinable.*

The following has been shown in [4].

Lemma. (a) *Every paracompact space is expandable.*

(b) *A space X is countably paracompact iff X is countably expandable.*

Theorem 2. *A space X is paracompact iff X is $B(LF, \lambda)$-refinable and expandable.*

Proof. The necessity is clear. To prove the sufficiency, assume that X is expandable and $B(LF, \lambda)$-refinable. Let $\mathcal{U}$ be an open cover of X, and $\mathcal{E}=\cup\{\mathcal{E}_\gamma : \gamma<\lambda\}$ a $B(LF, \lambda)$-refinement of $\mathcal{U}$. We use induction to construct a family $\mathcal{V}^*=\{\mathcal{V}_\gamma : \gamma<\lambda\}$ of collections of subsets of X satisfying

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(i) $\mathcal{V}_\gamma$ is a LF-open partial refinement of $\mathcal{U}$ for each $\gamma<\lambda$, and

(ii) $\cup\{\cup\mathcal{E}_\beta:\beta<\gamma\}\subset\cup\{\cup\mathcal{V}_\beta:\beta<\gamma\}$ for each $\gamma<\lambda$.

Let $\gamma<\lambda$ be fixed, and assume that collections $\mathcal{V}_\beta$ have been constructed such that conditions (i) and (ii) above are satisfied for all $\beta<\gamma$. Define $V^*=\cup\{\cup\mathcal{V}_\beta:\beta<\gamma\}$, and $\mathcal{F}_\gamma=\{E-V^*:E\in\mathcal{E}_\gamma\}$. Now $\mathcal{F}_\gamma$ is a LF-closed refinement of $\mathcal{U}$, and X is expandable; hence, $\mathcal{F}_\gamma$ has a LF-open expansion $\mathcal{V}_\gamma$ which partially refines $\mathcal{U}$. It should be clear that $\cup\{\cup\mathcal{E}_\beta:\beta<\gamma\}\subset\cup\{\cup\mathcal{V}_\beta:\beta<\gamma\}$, and our construction is complete. Now define $\mathcal{V}=\cup\{\mathcal{V}_\gamma:\gamma<\lambda\}$.

Since $\mathcal{E}=\cup\{\mathcal{E}_\gamma:\gamma<\lambda\}$ covers X, conditions (i) and (ii) above imply that $\mathcal{V}$ is a $\sigma-LF$-open refinement of $\mathcal{U}$. Now $\{\cup\mathcal{V}_\gamma:\gamma<\lambda\}$ is a countable open cover of X. By the lemma above, X is countably paracompact, and so $\{\cup\mathcal{V}_\gamma:\gamma<\lambda\}$ has a LF-open refinement $\{W_\gamma:\gamma<\lambda\}$ such that $W_\gamma\subset\cup\mathcal{V}_\gamma$ for each $\gamma<\lambda$. For each $\gamma<\lambda$, define $\mathcal{G}_\gamma=\{W_\gamma\cap V:V\in\mathcal{V}_\gamma\}$, and $\mathcal{G}=\cup\{\mathcal{G}_\gamma:\gamma<\lambda\}$. It is easy to see that $\mathcal{G}$ is a LF-open refinement of $\mathcal{U}$. Therefore, X is paracompact.

In [11] it was shown that, a space X is expandable iff X is discretely-θ-expandable and countably paracompact. Hence we have the following.

Corollary. *A space X is paracompact iff X is countably paracompact, discretely-θ-expandable, and $B(LF,\lambda)$-refinable.*

Corollary. *Let X be any countably paracompact, strong-CWSN space. Then the following are equivalent.*

(a) *X is paracompact.*

(b) *X is subparacompact.*

(c) *X is metacompact.*

(d) *X is θ-refinable.*

(e) *X is $B(D,\omega)$-refinable.*

(f) *X is weak $\bar{\theta}$-refinable.*

(g) *X is $B(D,\lambda)$-refinable.*

(h) *X is $B(LF,\lambda)$-refinable.*

Proof. Clearly, (a)→(b), (g)→(h), and it is shown in [9] that (e)→(f)→(g). By Theorem 1, we have (b)↔(c)↔(d)↔(e). Furthermore, (h)↔(a) follows from above, since every strong-$CWSN$ space is discretely-θ-expandable.

Theorem 3. *A countably metacompact space X is collectionwise normal iff every open cover of X, which has a $B(LF,\lambda)$-refinement, is a normal cover.*

Proof. In [8] it is shown that a space X is collectionwise normal iff every weak $\bar{\theta}$-cover of X is a normal cover. Sufficiency follows. Now assume that X is countably metacompact and collectionwise normal. Let $\mathcal{U}$ be an open cover of X which has a $B(LF,\lambda)$-refinement $\mathcal{E}=\cup\{\mathcal{E}_\gamma:\gamma<\lambda\}$. We will show that $\mathcal{U}$ has a LF-open refinement, which implies $\mathcal{U}$ is a normal cover. By transfinite induction we construct for every $\gamma<\lambda$, a family $\{\mathcal{H}(\gamma,n):n\in N\}$ of collections of subsets of X satisfying:

(i) $\mathcal{H}(\gamma, n)$ is a LF collection of cozero sets for each $n \in N$,

(ii) $\mathcal{H}(\gamma, n)$ partially refines $\mathcal{U}$ for each $n \in N$, and

(iii) $\cup \mathcal{F}_\gamma \subset H^*_\gamma = \cup\{\cup \mathcal{H}(\gamma, n): n \in N\}$, where $\mathcal{F}_\gamma = \{E - \cup\{H^*_\beta : \beta < \gamma\}: E \in \mathcal{E}_\gamma\}$.

For fixed $\gamma < \lambda$, assume $\mathcal{H}(\beta, n)$ has been constructed such that conditions (i)–(iii) above are satisfied for all $\beta < \gamma$. Let $T = X - \cup\{H^*_\beta : \beta < \gamma\}$. Now $\mathcal{F}_\gamma$ is a LF-closed partial refinement of $\mathcal{U}$ whose union is contained in the closed, countably metacompact subspace T. For each $n \in N$, define

$$\mathcal{S}(\gamma, n) = \{x : ord(x, \mathcal{F}_\gamma)\} \leq n\} \cap T,$$

and

$$\mathcal{S}_\gamma = \{\mathcal{S}(\gamma, n) : n \in N\}.$$

Now $\mathcal{S}_\gamma$ is a countable monotone open cover of the countably metacompact subspace T. Therefore $\mathcal{S}_\gamma$ has a closed shrink

$$\mathcal{K}_\gamma = \{K(\gamma, n) : n \in N\}$$

such that $K(\gamma, n) \subset \mathcal{S}(\gamma, n)$ for each $n \in N$.

For each $n \in N$, define

$$\mathcal{L}(\gamma, n) = \{F \cap K(\gamma, n) : F \in \mathcal{F}_\gamma\},$$

and

$$\mathcal{L}_\gamma = \cup\{\mathcal{L}(\gamma, n) : n \in N\}.$$

Since each member of $\mathcal{L}(\gamma, n)$ is contained in $\mathcal{S}(\gamma, n)$, it follows that $\mathcal{L}(\gamma, n)$ is an n-bded-LF collection of closed subsets of X; therefore, $\mathcal{L}(\gamma, n)$ must have a LF-cozero-expansion $\mathcal{H}(\gamma, n)$ for each $n \in N$, which partially refines $\mathcal{U}$. It is easy to see that $\{\mathcal{H}(\gamma, n) : n \in N\}$ satisfies conditions (i)–(iii) above, and our construction is complete.

Since $\mathcal{H}(\gamma, n)$ is a LF collection of cozero sets, $\cup \mathcal{H}(\gamma, n)$ must be a cozero set for every $\gamma < \lambda$ and $n \in N$; hence, $\mathcal{H}^* = \{\cup \mathcal{H}(\gamma, n) : \gamma < \lambda, n \in N\}$ is a countable cozero cover of X. Thus $\mathcal{H}^*$ has a LF-open refinement $\mathcal{W} = \{W(\gamma, n) : \gamma < \lambda, n \in N\}$ such that $W(\gamma, n) \subset \cup \mathcal{H}(\gamma, n)$ for every $\gamma < \lambda$, $n \in N$.

Define $\mathcal{V}(\gamma, n) = \{W(\gamma, n) \cap H : H \in \mathcal{H}(\gamma, n)\}$ for every $\gamma < \lambda$, $n \in N$, and $\mathcal{V} = \cup\{\mathcal{V}(\gamma, n) : \gamma < \lambda, n \in N\}$. It is easy to see that $\mathcal{V}$ is a LF-open refinement of $\mathcal{U}$, and hence $\mathcal{U}$ must be a normal cover of X.

Corollary. *A space X is paracompact iff X is collectionwise normal and $B(LF, \lambda)$-refinable.*

Proof. The necessity should be clear. Now assume that X is collectionwise normal and $B(LF, \lambda)$-refinable. From Theorem 3 of [5] it follows that X is countably metacompact. Therefore by Theorem 3 above, every open cover of X is normal and hence X is paracompact.

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