78. A Nonlinear Ergodic Theorem for Asymptotically Nonexpansive Mappings in Banach Spaces

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1. Introduction. Throughout this paper X denotes a uniformly convex real Banach space and C is a closed convex subset of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). The duality mapping J (multivalued) from X into X^* will be defined by $J(x)=\{x^* \in X^* : (x, x^*)=\|x\|^2 =\|x^*\|^2\}$ for $x \in X$. We say that X is (F) if the norm of X is Fréchet differentiable, i.e., for each $x \in X$ with $x \neq 0$, $\lim_{t\to 0} t^{-1}(\|x+ty\|-\|x\|)$ exists uniformly in $y \in B_1$, where $B_r=\{z \in X : \|z\| \leqq r\}$ for $r>0$. A mapping $T : C \to C$ is said to be *asymptotically nonexpansive* if for each $n=1, 2, \cdots$

(1.1) $$\|T^n x - T^n y\| \leqq (1+\alpha_n)\|x-y\| \quad \text{for any } x, y \in C,$$

where $\lim_{n\to\infty} \alpha_n=0$. In particular, if $\alpha_n=0$ for all $n \geqq 1$, T is said to be nonexpansive. The set of fixed points of T will be denoted by $F(T)$.

Throughout the rest of this paper let $T : C \to C$ be an asymptotically nonexpansive mapping satisfying (1.1).

A sequence $\{x_n\}_{n\geqq 0}$ in C is called an *almost-orbit* of T if

$$\lim_{n\to\infty} [\sup_{m\geqq 0} \|x_{n+m} - T^m x_n\|]=0.$$

A sequence $\{z_n\}$ in X is said to be *weakly almost convergent* to $z \in X$ if

$$w-\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} z_{k+i} = z$$

uniformly in $i \geqq 0$.

The purpose of this paper is to prove the following (nonlinear) mean ergodic theorem which is an extension of [3, Theorem 1] and [1, Corollary 2.1].

Theorem. *Let $\{x_n\}_{n\geqq 0}$ be an almost-orbit of T. If X is (F) and C is bounded, then $\{x_n\}$ is weakly almost convergent to the unique point of $F(T) \cap \operatorname{clco} \omega_w(\{x_n\})$, where $\omega_w(\{x_n\})$ denotes the set of weak subsequential limits of $\{x_n\}$, and $\operatorname{clco} E$ is the closed convex hull of E.*

2. Proof of Theorem. Throughout this section, we assume C is bounded. By Bruck's inequality [2, Theorem 2.1], we get

Lemma 1. *There exists a strictly increasing, continuous, convex function $\gamma : [0, \infty) \to [0, \infty)$ with $\gamma(0)=0$ such that*

$$\left\|T^k\left(\sum_{i=1}^n \lambda_i x_i\right) - \sum_{i=1}^n \lambda_i T^k x_i\right\| \leqq (1+\alpha_k)\gamma^{-1}\left(\max_{1\leqq i,j\leqq n}\left[\|x_i - x_j\| - \frac{1}{1+\alpha_k}\|T^k x_i - T^k x_j\|\right]\right)$$

for any k, $n \geqq 1$, *any* $\lambda_1, \cdots, \lambda_n \geqq 0$ *with* $\sum_{i=1}^n \lambda_i = 1$ *and any* $x_1, \cdots, x_n \in C$.

Hereafter, let γ be as in Lemma 1. Now, we can easily prove

Lemma 2. *Suppose that* $\{x_n\}_{n \geqq 0}$ *and* $\{y_n\}_{n \geqq 0}$ *are almost-orbits of* T. *Then* $\{\|x_n - y_n\|\}$ *converges as* $n \to \infty$.

We now put $D = $ diameter C and $M = \sup_{n \geqq 1}(1+\alpha_n)$.

Lemma 3. *Suppose that* $\{x_j^{(p)}\}_{j \geqq 1}$ $(p=1, 2, \cdots)$ *are almost-orbits of* T. *Then for any* $\varepsilon > 0$ *and* $n \geqq 1$ *there exist* $N_\varepsilon \geqq 1$ *and* $i_n(\varepsilon) \geqq 1$, *where* N_ε *is independent of* n, *such that* $\|T^k(\sum_{p=1}^n \lambda_p x_i^{(p)}) - \sum_{p=1}^n \lambda_p T^k x_i^{(p)}\| < \varepsilon$ *for any* $k \geqq N_\varepsilon$, *any* $i \geqq i_n(\varepsilon)$, *and any* $\lambda_1, \cdots, \lambda_n \geqq 0$ *with* $\sum_{p=1}^n \lambda_p = 1$.

Proof. For any $\varepsilon > 0$ choose $\delta > 0$ so that $\gamma^{-1}(\delta) < \varepsilon/M$. Then there exists $N_\varepsilon \geqq 1$ such that $\alpha_k < \delta/4D$ if $k \geqq N_\varepsilon$. Since $\{\|x_j^{(p)} - x_j^{(q)}\|\}_{j \geqq 1}$ converges as $j \to \infty$ by Lemma 2, for each p, $q \geqq 1$ there exists $i_0(\varepsilon, p, q) \geqq 1$ such that $|\|x_i^{(p)} - x_i^{(q)}\| - \|x_{i+k}^{(p)} - x_{i+k}^{(q)}\|| < \delta/4$ if $i \geqq i_0(\varepsilon, p, q)$ and $k \geqq 0$. Moreover, there are $i_1(\varepsilon, p) \geqq 1$ such that $a_i^{(p)} < \delta/4$ for all $i \geqq i_1(\varepsilon, p)$, where $a_i^{(p)} = \sup_{j \geqq 0}\|x_{i+j}^{(p)} - T^j x_i^{(p)}\|$. Put $i_n(\varepsilon) = \max\{i_0(\varepsilon, p, q), i_1(\varepsilon, p) : 1 \leqq p, q \leqq n\}$ for $n \geqq 1$. If $i \geqq i_n(\varepsilon)$ and $k \geqq N_\varepsilon$, then

$$\|x_i^{(p)} - x_i^{(q)}\| - \frac{1}{1+\alpha_k}\|T^k x_i^{(p)} - T^k x_i^{(q)}\|$$
$$\leqq |\|x_i^{(p)} - x_i^{(q)}\| - \|x_{i+k}^{(p)} - x_{i+k}^{(q)}\|| + a_i^{(p)} + a_i^{(q)} + \alpha_k\|x_i^{(p)} - x_i^{(q)}\| < \delta$$

for $1 \leqq p$, $q \leqq n$ and by Lemma 1, $\|T^k(\sum_{p=1}^n \lambda_p x_i^{(p)}) - \sum_{p=1}^n \lambda_p T^k x_i^{(p)}\| < \varepsilon$ for any $\lambda_1, \cdots, \lambda_n \geqq 0$ with $\sum_{p=1}^n \lambda_p = 1$. Q.E.D.

For any $\varepsilon > 0$ and $k \geqq 1$, we put $F_\varepsilon(T^k) = \{x \in C : \|T^k x - x\| \leqq \varepsilon\}$. Since C is bounded, $F(T) \neq \varnothing$. (For example, see [4, Proposition 2.3].)

Lemma 4. *Suppose that* $\{x_i\}_{i \geqq 0}$ *is an almost-orbit of* T. *Then for any* $\varepsilon > 0$ *there exists* $N_\varepsilon \geqq 1$ *such that for each* $k \geqq N_\varepsilon$, *there is* $N_k (= N_k(\varepsilon)) \geqq 1$ *satisfying* $(1/n)\sum_{i=0}^{n-1} x_{i+l} \in F_\varepsilon(T^k)$ *for all* $n \geqq N_k$ *and all* $l \geqq 0$.

Proof. Let $\varepsilon > 0$ be arbitrarily given and σ be the inverse function of $t \mapsto M\gamma^{-1}(3t) + t$. Put $\delta = \min\{\sigma(\varepsilon/3), (\varepsilon/3M'D)\}$ and $M' = M+1$. Choose $\eta > 0$ and $N_{1,\varepsilon} \geqq 1$ so that $\gamma^{-1}(\eta) < (\delta^2/2M)$ and $\alpha_k < \sigma(\varepsilon/3)/D$ if $k \geqq N_{1,\varepsilon}$. Furthermore, by Lemma 3, there exists $N_{2,\varepsilon} \geqq 1$ such that for any $p \geqq 1$ there is $i_p(\varepsilon) \geqq 1$ satisfying

$$\left\|T^k\left(\frac{1}{p}\sum_{j=0}^{p-1} x_{i+j+l}\right) - \frac{1}{p}\sum_{j=0}^{p-1} T^k x_{i+j+l}\right\| < \frac{\delta^2}{8} \tag{2.1}$$

for any $k \geqq N_{2,\varepsilon}$, any $i \geqq i_p(\varepsilon)$, and any $l \geqq 0$. Put $N_\varepsilon = \max(N_{1,\varepsilon}, N_{2,\varepsilon})$ and let $k \geqq N_\varepsilon$ be fixed. By Lemma 1 and the choice of δ, we get

$$\operatorname{clco} F_\delta(T^k) \subset F_{\varepsilon/3}(T^k). \tag{2.2}$$

Next, choose $p \geqq 1$ so that $(Dk/p) \leqq (\delta^2/2)$ and let p be fixed. Since $\{x_i\}_{i \geqq 0}$ is an almost-orbit of T, there exists $N \geqq 1$ such that if $m \geqq N$, $\sup_{q \geqq 0}\|x_{m+q} - T^q x_m\| < (\delta^2/8)$. Set $w_i = (1/p)\sum_{j=0}^{p-1} x_{i+j}$ for $i \geqq 0$. If $i \geqq i_p(\varepsilon) + N$, by (2.1),

$$\|w_{i+k+l} - T^k w_{i+l}\| \leqq \left\|\frac{1}{p}\sum_{j=0}^{p-1}(x_{i+j+k+l} - T^k x_{i+j+l})\right\|$$
$$+ \left\|\frac{1}{p}\sum_{j=0}^{p-1} T^k x_{i+j+l} - T^k\left(\frac{1}{p}\sum_{j=0}^{p-1} x_{i+j+l}\right)\right\| < \frac{\delta^2}{4}$$

for all $l\geqq 0$. Choose $N_3(k)\geqq i_p(\varepsilon)+N+1$ such that $(D(i_p(\varepsilon)+N)/n)<(\delta^2/4)$ for all $n\geqq N_3(k)$. If $n\geqq N_3(k)$, then

$$(2.3)\quad \frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+l}-T^k w_{i+l}\|\leqq \frac{1}{n}\sum_{i=0}^{n-1}\|w_{i+l}-w_{i+k+l}\|$$
$$+\frac{1}{n}\left(\sum_{i=0}^{i_p+N-1}+\sum_{i=i_p+N}^{n-1}\right)\|w_{i+k+l}-T^k w_{i+l}\|\leqq \frac{Dk}{p}+\frac{(i_p(\varepsilon)+N)D}{n}+\frac{\delta^2}{4}\leqq \delta^2$$

for all $l\geqq 0$, where $i_p=i_p(\varepsilon)$. Choose $N_4(k)\geqq 1$ so that $((p-1)D/2n)<(\varepsilon/3M')$ for all $n\geqq N_4(k)$. Put $N_k=\max(N_3(k), N_4(k))$ and let $n\geqq N_k$ be fixed and $l\geqq 0$. Set $A(k,n,l)=\{i\in Z:\ 0\leqq i\leqq n-1 \text{ and } \|w_{i+l}-T^k w_{i+l}\|\geqq\delta\}$ and $B(k,n,l)=\{0,1,\cdots,n-1\}\backslash A(k,n,l)$. By (2.3), $\# A(k,n,l)\leqq n\delta$ where $\#$ denotes cardinality. Let $f\in F(T)$. Then,

$$\frac{1}{n}\sum_{i=0}^{n-1}x_{i+l}=\frac{1}{n}\sum_{i=0}^{n-1}w_{i+l}+\frac{1}{np}\sum_{i=1}^{p-1}(p-i)(x_{i+l-1}-x_{i+l+n-1})$$
$$=\left[\frac{1}{n}(\# A(k,n,l))\cdot f+\frac{1}{n}\sum_{i\in B(k,n,l)}w_{i+l}\right]+\left[\frac{1}{n}\sum_{i\in A(k,n,l)}(w_{i+l}-f)\right]$$
$$+\frac{1}{np}\sum_{i=1}^{p-1}(p-i)(x_{i+l-1}-x_{i+l+n-1}).$$

The first term on the right side of the above equality is contained in $\operatorname{clco}F_\delta(T^k)$, and the rest term in $B_{2\varepsilon/3M'}$. By (2.2), we get $(1/n)\sum_{i=0}^{n-1}x_{i+l}\in F_\varepsilon(T^k)$ for all $l\geqq 0$. Q.E.D.

Lemma 5. *Let $\{x_n\}$ in C be such that $w-\lim_{n\to\infty}x_n=x$. Suppose that for any $\varepsilon>0$ there exists $N(\varepsilon)\geqq 1$ such that for $k\geqq N(\varepsilon)$ there is $N_k>0$ satisfying $\|T^k x_n-x_n\|<\varepsilon$ for all $n\geqq N_k$. Then $x\in F(T)$.*

Proof. We shall show that $\lim_{k\to\infty}\|T^k x-x\|=0$. For any $\varepsilon>0$ choose $\delta>0$ so that $\gamma^{-1}(\delta)<(\varepsilon/4M)$ and take $N_1(\varepsilon)\geqq 1$ such that $\alpha_k<(\delta/3D)$ for all $k\geqq N_1(\varepsilon)$. Put $\delta'=\min(\delta/3,\varepsilon/4)$. By the assumption, there exists $N(\varepsilon)\geqq 1$ such that for each $k\geqq N(\varepsilon)$ there is $N_k>0$ satisfying $\|T^k x_n-x_n\|<\delta'$ for all $n\geqq N_k$. Put $N_2(\varepsilon)=\max(N_1(\varepsilon), N(\varepsilon))$ and let $k\geqq N_2(\varepsilon)$ be arbitrarily fixed. Since $x\in\operatorname{clco}\{x_n\,|\,n\geqq N_k\}$, there exists a sequence $\{\sum_{i=1}^{l_n}\lambda_n^{(i)}x_{\psi_n(i)}\}\subset\operatorname{co}\{x_n\,|\,n\geqq N_k\}$ such that $\lim_{n\to\infty}\sum_{i=1}^{l_n}\lambda_n^{(i)}x_{\psi_n(i)}=x$. Therefore there is $N_3(k)\geqq 1$ such that $\|\sum_{i=1}^{l_n}\lambda_n^{(i)}x_{\psi_n(i)}-x\|<(\varepsilon/4M)$ for all $n\geqq N_3(k)$ and hence if $n\geqq N_3(k)$, $\|T^k x-T^k(\sum_{i=1}^{l_n}\lambda_n^{(i)}x_{\psi_n(i)})\|<(\varepsilon/4)$. On the other hand, by Lemma 1 and the choice of δ and k, we get $\|T^k(\sum_{i=1}^{l_n}\lambda_n^{(i)}x_{\psi_n(i)})-\sum_{i=1}^{l_n}\lambda_n^{(i)}T^k x_{\psi_n(i)}\|<(\varepsilon/4)$ for all $n\geqq 1$. Consequently,

$$\|T^k x-x\|\leqq\left\|T^k x-T^k\left(\sum_{i=1}^{l_n}\lambda_n^{(i)}x_{\psi_n(i)}\right)\right\|+\left\|T^k\left(\sum_{i=1}^{l_n}\lambda_n^{(i)}x_{\psi_n(i)}\right)-\sum_{i=1}^{l_n}\lambda_n^{(i)}T^k x_{\psi_n(i)}\right\|$$
$$+\left\|\sum_{i=1}^{l_n}\lambda_n^{(i)}(T^k x_{\psi_n(i)}-x_{\psi_n(i)})\right\|+\left\|\sum_{i=1}^{l_n}\lambda_n^{(i)}x_{\psi_n(i)}-x\right\|<\varepsilon,$$

where $n\geqq N_3(k)$. This shows that $\|T^k x-x\|<\varepsilon$ for $k\geqq N_2(\varepsilon)$. Q.E.D.

Lemma 6. *Suppose that X is (F) and $\{x_n\}$ is an almost-orbit of T. Then, the following hold:*

(i) *$\{(x_n, J(f-g))\}$ converges for every $f, g\in F(T)$.*

(ii) *$F(T)\cap\operatorname{clco}\omega_w(\{x_n\})$ is at most a singleton.*

Proof. Let $\lambda\in(0,1)$ and $f, g\in F(T)$. By Lemma 3, for any $\varepsilon>0$ there exist $N_\varepsilon\geqq 1$ and $i_2(\varepsilon)\geqq 1$ such that if $k\geqq N_\varepsilon$ and $n\geqq i_2(\varepsilon)$, then $\|T^k(\lambda x_n+(1-\lambda)f)-\lambda T^k x_n-(1-\lambda)f\|<\varepsilon$. Since $\|\lambda x_{n+m}+(1-\lambda)f-g\|\leqq\lambda\|x_{n+m}-T^m x_n\|+\|T^m(\lambda x_n+(1-\lambda)f)-\lambda T^m x_n-(1-\lambda)f\|+(1+\alpha_m)\|\lambda x_n+(1-\lambda)f-g\|\leqq\lambda\|x_{n+m}-T^m x_n\|+\varepsilon+(1+\alpha_m)\|\lambda x_n+(1-\lambda)f-g\|$ for $m\geqq N_\varepsilon$ and $n\geqq i_2(\varepsilon)$, $\{\|\lambda x_n+(1-\lambda)f-g\|\}$ converges. The rest of proof is the same as [5, Lemma 3.6]. Q.E.D.

Proof of Theorem. Let $\rho(n)$ be any sequence of nonnegative integers, and put $s_n=(1/n)\sum_{i=0}^{n-1}x_{i+\rho(n)}$. It suffices to show that $\{s_n\}$ converges weakly to a point of $F(T)\cap \mathrm{clco}\,\omega_w(\{x_n\})$. First, note $\omega_w(\{s_n\})\neq\emptyset$ because $\{s_n\}$ is bounded. Next, Lemmas 4 and 5 imply $\omega_w(\{s_n\})\subset F(T)$. Moreover $\omega_w(\{s_n\})\subset\bigcap_{i=0}^{\infty}\mathrm{clco}\,\{x_k: k\geqq i\}=\mathrm{clco}\,\omega_w(\{x_n\})$. Thus we have $\phi\neq\omega_w(\{s_n\})\subset F(T)\cap\mathrm{clco}\,\omega_w(\{x_n\})$. Combining this with Lemma 6-(ii), we obtain that $\omega_w(\{s_n\})$ is a singleton and is equal to $F(T)\cap\mathrm{clco}\,\omega_w(\{x_n\})$. Q.E.D.

Remarks. 1) The assumption "C is bounded" in Theorem may be replaced by "$F(T)\neq\emptyset$".

2) Similarly we can prove the mean ergodic theorem for an asymptotically nonexpansive semigroup.

In the same way as the proof of [1, Theorem 3.1], by virtue of Theorem, we get the following which improves upon [6, Corollary 3].

Corollary. *Suppose that X is (F) and $\{x_n\}$ is an almost-orbit of T. $\{x_n\}$ is weakly convergent to a fixed point of T if and only if $F(T)\neq\emptyset$ and $w-\lim_{n\to\infty}(x_n-x_{n+1})=0$.*

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