54. *A Discrepancy Problem with Applications to Linear Recurrences. II*

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This is continued from [0].

The following result gives an estimation for the discrepancy of a special s-dimensional sequence (x_n), $n=1,2,\cdots$. Let us recall the definition of the *discrepancy* $D_N(x_n)$. Generally speaking the discrepancy is a measure for the distribution behaviour of (x_n) modulo 1. More precisely put

$$A_N(x_n, I)=\text{card}\,\{n\leqq N : \{x_n\}\in I\}$$

for the number indices n such that the (componentwise) fractional part of x_n is contained in a given s-dimensional interval I. Then

$$D_N(x_n) := \sup_I \left|\frac{A_N(x_n, I)}{N}-|I|\right|,$$

where the supremum is taken over all s-dimensional subintervals I of $[0,1]^s$ with volume $|I|$. Thus, if $|I|\geqq 2D_N$, there exists an integer n with $1\leqq n\leqq N$, such that $\{x_n\}\in I$. If $D_N(x_n)$ tends to zero (for $N\to\infty$) then (x_n) is called *uniformly distributed* modulo 1 (cf. [6]).

Theorem 1. *Let $y_1,\cdots,y_s$ be a multiplicatively independent system of unimodular complex algebraic numbers and let θ_k be real numbers defined by*

$$y_k=e^{2\pi i\theta_k}\ (k=1,\cdots,s).$$

Set $\theta=(\theta_1,\cdots,\theta_s)$ and let $\omega=(\omega_1,\cdots,\omega_s)$ be an arbitrary s-tuple of real numbers. Then the discrepancy of the s-dimensional sequence $(x_n)=(n\theta+\omega)$ satisfies the estimate

$$D_N(x_n)\leqq N^{-\delta}$$

for any sufficiently large N, where $\delta(>0)$ depends only on the system $y_1,\cdots,y_s$.

Proof. Let m be an arbitrary positive integer. Then by the inequality of Erdös-Turán-Koksma (cf. [6], p. 116) we have

$$(9)\qquad D_N(x_n)\leqq c_s\left(\frac{1}{m}+\sum_{0<\|h\|\leqq m}\frac{1}{r(h)}\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i\langle h,x_n\rangle}\right|\right),$$

where c_s is a constant depending only on the dimension s, the first sum runs through all integral lattice points $h=(h_i,\cdots,h_s)\neq(0,\cdots,0)$ with

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$\|h\|=\max(|h_1|, \cdots, |h_s|)\leqq m$, $\langle\cdot,\cdot\rangle$ denotes the usual inner product in $\boldsymbol{R}^s$ and $r(h)$ is defined by

$$r(h)=\prod_{j=1}^{s}\max(|h_j|, 1).$$

Using the summation formula of geometric series we obtain

$$\left|\sum_{n=1}^{N} e^{2\pi i\langle h, x_n\rangle}\right| = \left|\sum_{n=1}^{N} e^{2\pi i\langle h, n\theta+\omega\rangle}\right| = \left|\sum_{n=1}^{N} e^{2\pi i n\langle h, \theta\rangle}\right|$$
$$=\left|\frac{e^{2\pi i N\langle h,\theta\rangle}-1}{e^{2\pi i\langle h,\theta\rangle}-1}\right| \leqq \frac{2}{|y_1^{h_1}\cdots y_s^{h_s}-1|}$$

with $y_k=e^{2\pi i\theta_k}$ $(k=1, \cdots, s)$. Thus we have by Lemma 3

$$\left|\sum_{n=1}^{N} e^{2\pi i\langle h, x_n\rangle}\right| \leqq 2\|h\|^{c_2} \tag{10}$$

for $\|h\|>n_0$ and a positive constant c_2 only depending on the system $y_1, \cdots, y_s$. Inserting (10) into (9) yields

$$D_N(x_n)\leqq c_s\left(\frac{1}{m}+\sum_{0\leqq\|h\|\leqq m}\frac{2}{r(h)}\cdot\frac{\|h\|^{c_2}}{N}\right)$$
$$=O\left(\frac{1}{m}+\frac{m^{c_2}(\log m)^{s-1}}{N}\right)=O\left(\frac{1}{m}+\frac{m^{c_3}}{N}\right)$$

for an arbitrary positive constant c_3 with $c_3>c_2$. Choosing

$$m=[N^{1/(c_3+1)}]+1$$

we obtain

$$D_N(x_n)=O\left(N^{-1/(c_3+1)}+\frac{N^{c_3/(c_3+1)}}{N}\right)=O(N^{-1/(c_3+1)})<N^{-\delta}$$

for sufficiently large N and for any δ with $0<\delta<(1/(c_2+1))$. Thus the proof of Theorem 1 is complete.

3. Main results. In this section we will apply the one-dimensional case of Theorem 1 in order to obtain bounds for the approximation of the characteristic roots by the quotients of subsequent values of second order linear recursive sequences.

Theorem 2. *For any non-degenerate second order linear recurrence R, for which $D<0$, there is a constant $c>0$ such that*

$$\left||\alpha|-\left|\frac{R_{n+1}}{R_n}\right|\right|<\frac{1}{n^c}$$

for infinitely many n.

Proof. By (1) and $|\alpha|=|\beta|$ we have

$$\left|\frac{R_{n+1}}{R_n}\right|=|\alpha|\cdot\left|\frac{\frac{a}{b}\left(\frac{\alpha}{\beta}\right)^{n+1}-1}{\frac{a}{b}\left(\frac{\alpha}{\beta}\right)^{n}-1}\right| \tag{11}$$

and by our notation

$$\arg\left(\frac{a}{b}\right)=2\omega\pi, \qquad \arg\left(\frac{\alpha}{\beta}\right)=2\theta\pi.$$

Let N be a given positive integer, and D_N be the discrepancy of the sequence

$(n\theta+\omega)$, $n=1,\cdots,N$. Then, by the definition of the discrepancy, we can choose an integer m with $1\leqq m\leqq N$ such that

$$\left|2\omega\pi+\arg\left(\frac{\alpha}{\beta}\right)^m-(2\pi k+\pi-\theta\pi)\right|\leqq 2D_N \tag{12}$$

and

$$\left|2\omega\pi+\arg\left(\frac{\alpha}{\beta}\right)^{m+1}-(2\pi k+\pi+\theta\pi)\right|\leqq 2D_N, \tag{13}$$

where k is a suitable integer. Furthermore we have

$$\frac{a}{b}\left(\frac{\alpha}{\beta}\right)^m=z+\varepsilon_1 \quad\text{and}\quad \frac{a}{b}\left(\frac{\alpha}{\beta}\right)^{m+1}=\bar{z}+\varepsilon_2 \tag{14}$$

for some complex numbers z, ε_1, ε_2 with $|\varepsilon_j|=O(D_N)$ for $j=1,2$. Hence we obtain by (11), (12), (13), and (14)

$$\left|\frac{R_{m+1}}{R_m}\right|=|\alpha|\cdot\left|\frac{z-1+\varepsilon_2}{\bar{z}-1+\varepsilon_1}\right|=|\alpha|\cdot\Big(1+O(D_N)\Big).$$

Thus, by the one-dimensional case of Theorem 1 we have

$$\left||\alpha|-\left|\frac{R_{m+1}}{R_m}\right|\right|=O(D_N)=O(N^{-\delta})<\frac{1}{m^c}$$

for any positive constant $c<\delta$. This completes the proof.

Our final result gives a lower bound for the approximation by the quotients of subsequent terms of second order linear recursive sequences.

Theorem 3. *For any non-degenerate second order linear recurrence R, for which $D<0$, there is a constant $c'>0$ such that*

$$\left||\alpha|-\left|\frac{R_{n+1}}{R_n}\right|\right|>\frac{1}{n^{c'}}$$

for all sufficiently large n.

Proof. Using the notations of the introduction, let z be a complex number defined by

$$z=e^{(\pi-\pi\theta)i}. \tag{15}$$

So we can write

$$e^{2\pi n\theta i+2\pi\omega i}=z\cdot e^{\lambda i}, \tag{16}$$

where $0\leqq\lambda<2\pi$ and

$$\begin{aligned}\lambda&=\pi(2n+1)\theta+2\pi\omega-\pi-2k\pi\\&=(2n+1)\cdot\arg(\alpha)+\arg(a/b)-(2k+1)\cdot\arg(-1)\\&=|(2n+1)\cdot\log\alpha-(2n+1)\cdot\log|\alpha|+\log(a/b)-(2k+1)\cdot\log(-1)|\end{aligned}$$

with some non-negative integer $k<n+1$. But $\lambda=0$ holds only for at most one value of n since otherwise α/β would be a root of unity. Thus $\lambda\neq 0$ if n is sufficiently large; furthermore α, $|\alpha|$, a/b, and -1 are algebraic numbers of degree at most 4 and so by Lemma 1 we have

$$|\lambda|>n^{-c_4}$$

with a constant c_4 depending on the parameters of the sequence R.

We can prove similarly the inequalities

$$|\pi-\lambda|>n^{-c_4}$$

and

$$|2\pi-\lambda|>n^{-c_4}.$$

From these

(17) $$|\mathrm{Im}\,(e^{\lambda i})|>n^{-c_5}$$

follows with any c_5 greater than c_4 if n is sufficiently large.

By (4), (15), and (16) we get

$$\left|\frac{R_{n+1}}{R_n}\right|=|\alpha|\cdot\left|\frac{\bar{z}\cdot e^{\lambda i}-1}{z\cdot e^{\lambda i}-1}\right|.$$

Now, since $|z|=|e^{\lambda i}|=1$, by (17), using Lemma 2 with $w=e^{\lambda i}$,

$$\left||\alpha|-\left|\frac{R_{n+1}}{R_n}\right|\right|>|\alpha|\cdot c_1\cdot|\mathrm{Im}\,(e^{\lambda i})|>n^{-c'}$$

follows for any sufficiently large n and for any c' greater than c_5. This proves Theorem 3.

Reference*)

[0] P. Kiss and R. F. Tichy: A discrepancy problem with applications to linear recurrences. I. Proc. Japan Acad., 65A, 135–138 (1989).

*) For other references, see [0].