50. *Regular Near-rings without Non-zero Nilpotent Elements*

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1. Introduction. In ring theory, it is well known that regular rings without non-zero nilpotent elements are characterized in terms of quasi-ideals, that is, the following conditions on a ring R are equivalent:

(a) R is regular and has no non-zero nilpotent elements.

(b) Every quasi-ideal of R is idempotent.

(c) For any two quasi-ideals Q_1, Q_2 of R, $Q_1 \cap Q_2 = Q_1Q_2$.

See [4, Theorem 11.5].

The purpose of this note is to characterize regular zero-symmetric near-rings without non-zero nilpotent elements, in terms of quasi-ideals. The analogy between the regular rings and near-rings without non-zero nilpotent elements is not complete.

For the basic terminology and notation we refer to [3].

2. Preliminaries. Let N be a near-ring, which always means right zero-symmetric one throughout this note.

If A, B and C are three non-empty subsets of N, then AB (ABC) denotes the set of all finite sums of the form $\sum a_kb_k$ with $a_k \in A$, $b_k \in B$ ($\sum a_kb_kc_k$ with $a_k \in A$, $b_k \in B$, $c_k \in C$). Note that $ABC = (AB)C \subseteq A(BC)$ in general.

A right N-subgroup (left N-subgroup) of N is a subgroup H of $(N, +)$ such that $HN \subseteq H$ ($NH \subseteq H$). For every subgroup H of $(N, +)$, HN is a right N-subgroup of N.

A quasi-ideal of N is a subgroup Q of $(N, +)$ such that $QN \cap QN \subseteq Q$ (see [5, Proposition 3]). Right N-subgroups and left N-subgroups are quasi-ideals. The intersection of a family of quasi-ideals is again a quasi-ideal.

A near-ring N is called regular, if for every element n of N there exists an element x in N such that $nxn = n$.

Lemma 1. *If a near-ring N is regular, then every quasi-ideal Q of N has the form $QNQ = Q$.*

Proof. Let Q be a quasi-ideal of N, that is, $QN \cap NQ \subseteq Q$. By the regularity of N, $Q \subseteq QNQ$. Moreover we have $QNQ \subseteq QN$ and $QNQ \subseteq NQ$. Hence it follows that

$$Q \subseteq QNQ \subseteq QN \cap NQ \subseteq Q.$$

Thus $Q = QNQ$.

Now we state here some known results which will be used later.

Lemma 2 (Ligh-Utumi [2]). *If N is a regular near-ring without non-zero nilpotent elements, then for any two elements n, m of N there exists an element x in N such that $nm=nxn$.*

A near-ring N is said to be an S-near-ring, if $n \in Nn$ for every element n of N. A near-ring N is called left bipotent, if $Nn=Nn^2$ for every element n of N.

Lemma 3 (Jat-Choudhary [1]). *If N is a left bipotent S-near-ring, then N is regular and has no non-zero nilpotent elements.*

3. Main result. First of all, we consider some difference between the regular rings and near-rings without non-zero nilpotent elements.

In case R is a near-ring, neither the condition (b) nor (c) in Introduction is equivalent to the condition (a), as illustrated by the following examples.

Example 1. Let $M=\{0, 1, 2, 3\}$ be the near-ring due to [3, Near-rings of low order (D-5)] defined by the tables

$+$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	1	0
2	0	2	2	0
3	0	3	3	0

.

Then M has three quasi-ideals: $\{0\}$, $\{0, 2\}$ and M. All of them are idempotent. But M is not regular, since $3m3=0$ for every element m of M. Thus (b) does not imply (a).

Example 2. Let $V=\{0, 1, 2, 3\}$ be the near-ring due to [3, Near-rings of low order (E-1)] defined by the tables

$+$	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	2	2	2
3	0	3	3	3

.

Then V is regular and has no non-zero nilpotent elements. But, for quasi-ideals $Q_1=\{0, 1\}$, $Q_2=\{0, 2\}$ of V, we have $Q_1 \cap Q_2 \neq Q_1Q_2$. Thus (a) does not imply (c).

Now we state the main result of this note.

Theorem. *The following conditions on a zero-symmetric near-ring N are equivalent:*

(1) *N is regular and has no non-zero nilpotent elements.*

(2) *N is an S-near-ring, and every quasi-ideal of N is an idempotent right N-subgroup of N.*

(3) *N is an S-near-ring, and for any two left N-subgroups L_1, L_2 of N, $L_1 \cap L_2 = L_1L_2$.*

Proof. (1)$\Rightarrow$(2): Clearly every regular near-ring is an S-near-ring. Let Q be a quasi-ideal of N. Any element x of QN has the form $x=\sum q_k n_k$ with $q_k \in Q$, $n_k \in N$. For each k, by Lemma 2, there is an element x_k in N such that $q_k n_k = q_k x_k q_k$. So $x \in QNQ$ and $QN \subseteq QNQ$. Therefore it follows from Lemma 1 that

$$Q = QNQ \subseteq QN \subseteq QNQ,$$

that is, $Q=QN$. Hence $QQ=(QN)Q=QNQ=Q$. Thus Q is an idempotent right N-subgroup of N.

(2)$\Rightarrow$(3): It is easy to see that the relation $RL \subseteq R \cap L$ always holds for every right N-subgroup R and left N-subgroup L of N. Now let L_1, L_2 be left N-subgroups of N. Since L_1, L_2 and $L_1 \cap L_2$ are quasi-ideals of N, it follows from the assumption (2) that

$$L_1 L_2 \subseteq L_1 \cap L_2 = (L_1 \cap L_2)^2 \subseteq L_1 L_2,$$

that is, $L_1 L_2 = L_1 \cap L_2$.

(3)$\Rightarrow$(1): Let n be any element of N. Since Nn and N are left N-subgroups of N, it follows from the assumption (3) that

$$Nn = Nn \cap Nn = (Nn)(Nn) \quad \text{and} \quad Nn = Nn \cap N = NnN.$$

So we get $Nn=(Nn)(Nn)=(NnN)n=Nn^2$. Thus N is a left bipotent S-near-ring. Hence, by Lemma 3, N is regular and has no non-zero nilpotent elements.

4. Remarks. In the condition (2) of Theorem, there are the following three properties of a zero-symmetric near-ring N:

(i) N is an S-near-ring.

(ii) Every quasi-ideal of N is idempotent.

(iii) Every quasi-ideal of N is a right N-subgroup of N.

Remark 1. The property (i) does not follow from the properties (ii) and (iii). Consider the near-ring M in Example 1. Then M has the properties (ii) and (iii). But M does not have the property (i), since $3 \notin M3$.

This example also shows that the converse of Lemma 1 does not hold.

Remark 2. The property (ii) does not follow from the properties (i) and (iii). Let $K=\{0, 1, 2, 3\}$ be the near-ring due to [3, Near-rings of low order (D-10)], whose addition coincides with that of M in Example 1 and whose multiplication is defined by the table

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	1
2	0	2	0	2
3	0	3	2	3

Then K has three quasi-ideals: $\{0\}$, $\{0, 2\}$ and K. It is easy to see that K has the properties (i) and (iii). But K does not have the property (ii), since $\{0, 2\}^2 \neq \{0, 2\}$.

Remark 3. It is also natural to ask whether the property (iii) follows from the properties (i) and (ii). This question is still open. However, there exists an S-near-ring N with a quasi-ideal which is idempotent but not a right N-subgroup of N.

Let $W=\{0,1,2,3\}$ be the near-ring due to [3, Near-rings of low order (E-13)], whose addition coincides with that of V in Example 2 and whose multiplication is defined by the table

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	0	0	0
3	0	1	2	3

Then W is an S-near-ring. The quasi-ideal $\{0,1\}$ of W is idempotent but not a right W-subgroup of W, since $\{0,1\}\,W=W$.

References

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