36. Some Results on Asymptotic Stability by Extending Matrosov's Theorems

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1. Introduction. In this paper we present asymptotic stability theorems for ordinary differential equations by extending Matrosov's theorems [4].

Let us consider the following ordinary differential equation:

$$\dot{x}=X(t,x), \qquad (X(t,0)\equiv 0), \tag{1}$$

where $X: \Gamma\to \boldsymbol{R}^n$ is a continuous function, $\Gamma=\boldsymbol{R}^+\times D$, $\boldsymbol{R}^+=[0,+\infty)$, and D is a domain in $\boldsymbol{R}^n$ satisfying $0\in D$.

Generalization of Liapunov's asymptotic stability theorem is considered by Barbashin and Krasovskiĭ (see [2] and [5]), Matrosov [4], LaSalle [3], Hatvani [1], Wada and Yamamoto [7] and etc. These results include the condition that the total derivative of a Liapunov function computed along the solutions of (1) is only negative semi-definite.

In the present paper, by extending Theorems 1.2 and 1.4 in [4], we establish theorems for (globally) asymptotic stability, (globally) equi-asymptotic stability and (globally) uniformly asymptotic stability as well as uniform stability of the zero solution of (1). In Theorems 1.2 and 1.4 of [4], Matrosov assumed that the function X, its partial derivatives $\partial X/\partial t$, $\partial X/\partial x_i$ $(i=1,2,\cdots,n)$, and the first and second partial derivatives of a Liapunov function V, that is, $\partial V/\partial t$, $\partial V/\partial x_i$, $\partial^2 V/\partial t^2$, $\partial^2 V/\partial t\partial x_i$, $\partial^2 V/\partial x_i\partial x_j$ $(i,j=1,2,\cdots,n)$, are continuous and bounded. In the foregoing paper [6], we extended Theorem 1.2 in [4] and gave uniform asymptotic stability theorems in which we generalized the above mentioned assumptions by Matrosov. Our resulting theorems in the present paper includes more useful conditions than the preceding paper's.

2. Theorems. For $\varepsilon>0$, B_ε is defined by $B_\varepsilon=\{x\in \boldsymbol{R}^n : \|x\|<\varepsilon\}$ and for $\alpha_1>\alpha_2>0$, the set $\Lambda(\alpha_1,\alpha_2)$ is defined by $\Lambda(\alpha_1,\alpha_2)=\{x\in\boldsymbol{R}^n : \alpha_1\le\|x\|\le\alpha_2\}$, where $\|x\|$ denotes the Euclidean norm of $x\in\boldsymbol{R}^n$. Let $C[A,E]$ be the family of all continuous functions from a set A into a set E. A function $a(\cdot)$ is called a function of class $\mathcal{K}$, i.e., $a\in\mathcal{K}$, if $a\in C[\boldsymbol{R}^+,\boldsymbol{R}^+]$ is a strictly increasing function with $a(0)=0$. The positive part $[x]_+$ of $x\in\boldsymbol{R}$ is defined by $[x]_+=\max\{0,x\}$, and the negative part $[x]_-$ of x is defined by $[x]_-=\max\{0,-x\}$. For a function $V\in C[\Gamma,\boldsymbol{R}]$ which is locally Lipschitzian in x, the total derivative $\dot{V}_{(1)}(t,x)$ of V with respect to (1) is defined by

$$\dot{V}_{(1)}(t,x)=\limsup_{h\to+0}\frac{1}{h}\{V(t+h,x+hX(t,x))-V(t,x)\},$$

in Γ. Let $x(\cdot\,;t_0,x_0)$ be a solution of (1) passing through a point $(t_0,x_0)\in\Gamma$.

Theorem 1. *Suppose that there exists a function $V\in C[\Gamma,\boldsymbol{R}^+]$ which is locally Lipschitzian in x. For some $h>0$ satisfying $\bar{B}_h\subset D$ and for any $\alpha>0$ $(\alpha<h)$, there exists a continuously differentiable function $W:\boldsymbol{R}^+\times\Lambda(\alpha,h)\to\boldsymbol{R}$ such that the following conditions hold.*

(i) *There exist functions $a,b\in\mathcal{K}$ such that for any $(t,x)\in\Gamma$,*

$$a(\|x\|)\le V(t,x)\le b(\|x\|).$$

(ii) *For any $(t,x)\in\Gamma$, $\dot{V}_{(1)}(t,x)\le 0$, and there exists a function $U\in C[\boldsymbol{R}^+\times\Lambda(\alpha,h),\boldsymbol{R}^+]$ which is locally Lipschitzian in x and absolutely continuous with respect to t such that for any $(t,x)\in\boldsymbol{R}^+\times\Lambda(\alpha,h)$,*

$$\dot{V}_{(1)}(t,x)\le -U(t,x).$$

(iii) *There exist constants $r=r(\alpha)>0$ and $L=L(\alpha)>0$ such that for any $(t,x)\in E_r=\{(t,x)\in\boldsymbol{R}^+\times\Lambda(\alpha,h): U(t,x)<r\}$,*

$$|W(t,x)|\le L.$$

(iv) *There exists a function $\xi\in C[\boldsymbol{R}^+,\boldsymbol{R}^+\backslash\{0\}]$ satisfying $\int_0^{+\infty}\xi(t)dt=+\infty$ such that for any $(t,x)\in E_r$,*

$$|\dot{W}_{(1)}(t,x)|\ge\xi(t).$$

(v) *There exists a constant $c_0=c_0(\alpha)>0$ such that either the following* (a) *or* (b) *is satisfied.*

(a) *For any $(t,x)\in\{(t,x)\in\boldsymbol{R}^+\times\Lambda(\alpha,h):(r/2)<U(t,x)<r\}$,*

$$[\dot{U}_{(1)}(t,x)]_+\le c_0.$$

(b) *For any $(t,x)\in\{(t,x)\in\boldsymbol{R}^+\times\Lambda(\alpha,h):(r/2)<U(t,x)<r\}$,*

$$[\dot{U}_{(1)}(t,x)]_-\le c_0.$$

Then the zero solution of (1) *is uniformly stable and attractive, and therefore it is asymptotically stable.*

Corollary 1. *In Theorem* 1, *let $D=\boldsymbol{R}^n$, that is, $\Gamma=\boldsymbol{R}^+\times\boldsymbol{R}^n$, and suppose that the function $a(\cdot)$ in* (i) *satisfies $a(\lambda)\to+\infty$ as $\lambda\to+\infty$. Further, suppose that for any constants h and α satisfying $0<\alpha<h$, condition* (ii) *holds and there exists a continuously differentiable function $W:\boldsymbol{R}^+\times\Lambda(\alpha,h)\to\boldsymbol{R}$ such that conditions* (iii)-(v) *hold. Then the zero solution of* (1) *is globally asymptotically stable.*

Notice that in Theorem 1, the constants r, L, c_0 and the function ξ depend only on α, but in Corollary 1, they depend on both α and h.

Theorem 2. *In addition to the assumptions of Theorem* 1, *suppose that the following condition* (vi) *holds.*

(vi) *Every solution of* (1) *passing through a point in Γ is unique to the right.*

Then the zero solution of (1) *is equiasymptotically stable.*

If, in addition to the assumptions of Corollary 1, *condition* (vi) *holds, then the zero solution of* (1) *is globally equiasymptotically stable.*

Theorem 2 is one of the extensions of Theorem 1.4 in [4]. The following Theorem 3 is obtained by extending Theorem 1.2 in [4].

Theorem 3. *In the assumptions of Theorem* 1, *suppose that condition* (iv) *is replaced by the following* (iv′).

(iv′) *There exists a constant* $\xi_0=\xi_0(\alpha)>0$ *such that for any* $(t, x) \in E_r=\{(t, x) \in \boldsymbol{R}^+ \times \Lambda(\alpha, h): U(t, x)<r\}$,

$$|\dot{W}_{(1)}(t, x)| \geq \xi_0.$$

Then the zero solution of (1) *is uniformly asymptotically stable.*

If, in the assumptions of Corollary 1, *condition* (iv) *is replaced by* (iv′), *then the zero solution of* (1) *is globally uniformly asymptotically stable.*

3. Proofs. *Proof of Theorem* 1. (i) and (ii) imply that the zero solution of (1) is uniformly stable, that is, for any $\varepsilon>0$, there exists a constant $\delta=\delta(\varepsilon)>0$ such that for every $t^* \geq 0$, $x^* \in B_\delta$, any solution $x(\cdot\,; t^*, x^*)$ of (1) and any $t \geq t^*$, $\|x(t; t^*, x^*)\|<\varepsilon$. Hence, there exists a constant $\eta>0$ such that for every $t_0 \geq 0$, $x_0 \in B_\eta$, any solution $x(\cdot) \equiv x(\cdot\,; t_0, x_0)$ of (1) and any $t \geq t_0$, $\|x(t)\|<h$.

Now we show that there exists $T=T(t_0, \varepsilon, x_0, x(\cdot))>0$ such that $\|x(t_0+T)\|<\delta$. Suppose that it is not true, that is, for any $t \geq t_0$, $\|x(t)\| \geq \delta$. Let $\alpha=\delta$. Then $x(t) \in \Lambda(\alpha, h)$ for $t \geq t_0$, and there exist constants $r>0$, $L>0$, $c_0>0$ and continuous functions U, ξ satisfying conditions (ii)-(v).

Using (i) and (ii), we can show that there exists a sequence $\{\tau_i\} \in \boldsymbol{R}^+$ such that $\tau_i \to +\infty$ as $i \to +\infty$ and for any $i \in \boldsymbol{N}$, $U(\tau_i, x(\tau_i))<(r/2)$. From (iii), (iv) and the continuity of $\dot{W}_{(1)}(\cdot, x(\cdot))$, we can also show that there exists a sequence $\{\tau_i'\} \in \boldsymbol{R}^+$ such that $\tau_i' \to +\infty$ as $i \to +\infty$ and for any $i \in \boldsymbol{N}$, $U(\tau_i', x(\tau_i')) \geq r$. Thus, from the continuity of $U(\cdot, x(\cdot))$, we can choose sequences $\{t_i\}$ and $\{t_i'\}$ such that for any $i \in \boldsymbol{N}$, $t_0 \leq t_1 < t_1' < \cdots < t_i < t_i' < t_{i+1} < \cdots$, $t_i \to +\infty$ as $i \to +\infty$, $U(t_i, x(t_i))=(r/2)$, $U(t_i', x(t_i'))=r$, (or $U(t_i, x(t_i))=r$, $U(t_i', x(t_i'))=(r/2)$ for any $i \in \boldsymbol{N}$) and $(r/2)<U(t, x(t))<r$ in (t_i, t_i'). Then condition (a) in (v) implies that for any $i \in \boldsymbol{N}$,

$$\frac{r}{2}=U(t_i', x(t_i'))-U(t_i, x(t_i))=\int_{t_i}^{t_i'} \dot{U}_{(1)}(t, x(t))dt$$
$$\leq \int_{t_i}^{t_i'} [\dot{U}_{(1)}(t, x(t))]_+ dt \leq c_0(t_i'-t_i).$$

Thus we have

$$t_i'-t_i \geq \frac{r}{2c_0}. \tag{2}$$

(If $U(t_i, x(t_i))=r$, $U(t_i', x(t_i'))=(r/2)$ for any $i \in \boldsymbol{N}$, then from condition (b) in (v), (2) is also obtained.) Hence it follows from (i) and (ii) that for any $k \in \boldsymbol{N}$,

$$\begin{aligned} -V(t_1, x(t_1)) &\leq V(t_k', x(t_k'))-V(t_1, x(t_1)) \\ &\leq \int_{t_1}^{t_k'} \dot{V}_{(1)}(s, x(s))ds \leq -\sum_{i=1}^{k} \int_{t_i}^{t_i'} U(s, x(s))ds \\ &\leq -\sum_{i=1}^{k} \frac{r}{2}(t_i'-t_i) \leq -\frac{r^2}{4c_0}k. \end{aligned} \tag{3}$$

Since there exists $k \in N$ such that the right hand side of (3) is less than the left hand side, this is a contradiction. Thus, there exists $T=T(t_0, \varepsilon, x_0, x(\cdot))>0$ such that $\|x(t_0+T)\|<\delta$, and it follows from the definition of δ that for any $t \geq t_0+T$, $\|x(t)\|<\varepsilon$. This implies that the zero solution of (1) is attractive, and therefore it is asymptotically stable. Q.E.D.

The proofs of Theorems 2 and 3 and the more detailed proof of Theorem 1 will be published later.

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