24. On Some Inequalities in the Theory of Uniform Distribution. I

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In this note, we present two inequalities for the supremum norm and the oscillation of a function satisfying a one-sided Lipschitz condition on the interval $E=[0,1]$ and having equal values at the end points. As special cases of them we obtain two estimates for the φ-discrepancy of a sequence of real numbers, with respect to a distribution function satisfying a Lipschitz condition on E. The results generalize some inequalities of LeVeque [3], Yurinskii [12], Niederreiter ([5], [6]), and Proinov ([7], [8]).

1. Definition 1. A real-valued function f is said to satisfy the *right Lipschitz condition* on E with a positive constant L if

$$f(x)-f(y)\leqq L(x-y) \qquad \text{for } x, y\in E \text{ with } x>y. \tag{1}$$

Analogously, f is said to satisfy the left Lipschitz condition if

$$f(x)-f(y)\geqq -L(x-y) \qquad \text{for } x, y\in E \text{ with } x>y. \tag{2}$$

The function f is said to satisfy the *one-sided Lipschitz condition* on E with constant L if either (1) or (2) holds.

It is easy to prove that if a function satisfies a one-sided Lipschitz condition on E, then it is a function of bounded variation on E. For a bounded function f on E, we denote by $\|f\|$ and $[f]$ its supremum norm and its oscillation, respectively.

Theorem 1. *Let a function f satisfy the one-sided Lipschitz condition on E with constant L, and let $f(0)=f(1)$ and $\|f\|\leqq L$. Then for any nondecreasing nonnegative function φ on $[0,\infty)$,*

$$F(\|f\|)\leqq L\int_0^1 \varphi(|f(x)|)dx \tag{3}$$

and

$$2F\left(\frac{1}{2}[f]\right)\leqq L\int_0^1 \varphi(|f(x)|)dx, \tag{4}$$

where the function F is defined on $[0,\infty)$ by

$$F(x)=\int_0^x \varphi(t)dt. \tag{5}$$

Proof. We shall prove only (4) since (3) can similarly be proved. We may assume that f satisfies a left Lipschitz condition since the other case follows immediately from this one (replacing f by $-f$). Now we extend f on $\boldsymbol{R}$ with period 1. Then it is easy to prove that the extended function f satisfies the left Lipschitz condition on the whole real line $\boldsymbol{R}$ with constant L. First we shall prove that the inequality

$$2F\left(\frac{1}{2}|f(\alpha)-f(\beta)|\right)\leqq L\int_0^1\varphi(|f(x)|)dx \tag{6}$$

holds for all $\alpha, \beta \in \boldsymbol{R}$. With no loss of generality, we may assume that $f(\alpha) > f(\beta)$ and $\alpha < \beta \leqq \alpha+1$. There are three possible cases: $f(\alpha)>0>f(\beta)$, $f(\alpha)>f(\beta)\geqq 0$, and $f(\beta)<f(\alpha)\leqq 0$.

Let $f(\alpha)>0>f(\beta)$. Since f satisfies the left Lipschitz condition on $\boldsymbol{R}$ with constant L, we have

$$f(x)\geqq f(\alpha)-L(x-\alpha)>0 \qquad \text{for } x\in(\alpha, \alpha'), \tag{7}$$

where $\alpha'=\alpha+f(\alpha)/L$. Therefore

$$\begin{aligned}&\int_\alpha^{\alpha'}\varphi(|f(x)|)dx \\ &\qquad \geqq\int_\alpha^{\alpha'}\varphi(f(\alpha)-L(x-\alpha))dx=(1/L)F(f(\alpha)).\end{aligned} \tag{8}$$

Analogously, we prove that

$$f(x)\leqq f(\beta)+L(x-\beta)<0 \qquad \text{for } x\in(\beta', \beta), \tag{9}$$

and

$$\int_{\beta'}^{\beta}\varphi(|f(x)|)dx\geqq(1/L)F(-f(\beta)), \tag{10}$$

where $\beta'=\beta-f(\alpha)/L$. From (7) and (9), we conclude that the intersection of the intervals (α, α') and (β', β) is an empty set. It is well known that if φ is nondecreasing on $[0, \infty)$, then F is a convex function on this interval. Hence, from (8) and (10), we deduce

$$\begin{aligned}2F\left(\frac{1}{2}|f(\alpha)-f(\beta)|\right)&=2F\left(\frac{1}{2}(f(\alpha)-f(\beta))\right)\\ &\leqq F(f(\alpha))+F(-f(\beta))\\ &\leqq L\left(\int_\alpha^{\alpha'}\varphi(|f(x)|)dx+\int_{\beta'}^{\beta}\varphi(|f(x)|)dx\right)\\ &\leqq L\int_\alpha^{\alpha+1}\varphi(|f(x)|)dx=L\int_0^1\varphi(|f(x)|)dx,\end{aligned}$$

and so (6) is proved in the first case.

Now let $f(\alpha)>f(\beta)\geqq 0$. We have $\alpha'=\alpha+f(\alpha)/L\leqq\alpha+1$ since $\|f\|\leqq L$. Hence, we obtain from (8).

$$\begin{aligned}F(|f(\alpha)-f(\beta)|)&\leqq F(f(\alpha))\\ &\leqq L\int_\alpha^{\alpha'}\varphi(|f(x)|)dx\\ &\leqq L\int_\alpha^{\alpha+1}\varphi(|f(x)|)dx=L\int_0^1\varphi(|f(x)|)dx.\end{aligned}$$

From this, we again arrive at (6) since $2F((1/2)x)\leqq F(x)$ for $x\geqq 0$.

In the case $f(\beta)<f(\alpha)\leqq 0$, the inequality (6) can be proved in the same way as in the previous case.

Now taking supremum on the left-hand side of (6) over all $\alpha, \beta\in E$, and taking into account that f is continuous and nondecreasing, we get the desired inequality (4). Q.E.D.

Corollary 1. *Let a function f satisfy the one-sided Lipschitz condition on E with constant L. Suppose also that*

$$f(0)=f(1) \quad \text{and} \quad \int_0^1 f(x)dx=0.$$

Then for any nondecreasing nonnegative function φ *on* $[0, \infty)$, *we have* (4).

Proof. According to Theorem 1, it is sufficient to prove that $\|f\| \leqq L$. Let us assume that $\|f\| > L$. Then there exists $\alpha \in E$ which satisfies either $f(\alpha) > L$ or $f(\alpha) < -L$. We treat only the first alternative, the second one being almost identical. With no loss of generality, we can suppose that f satisfies a left Lipschitz condition. Now extend f on $\boldsymbol{R}$ with period 1. Then from (7) and the inequality $f(\alpha) > L$, we conclude that $f(x) \geqq l(x) > 0$ for $x \in (\alpha, \alpha+1)$, where l is a linear function. Therefore, we have

$$\int_0^1 f(x)dx = \int_\alpha^{\alpha+1} f(x)dx \geqq \int_\alpha^{\alpha+1} l(x)dx > 0,$$

which is a contradiction. Q.E.D.

Corollary 2. *Let a function* f *satisfy the one-sided Lipschitz condition on* E *with constant* L, *and let* $f(0)=f(1)$. *Then*

$$[f] \leqq ((6L/\pi^2) \textstyle\sum_{h=1}^{\infty} (1/h^2) |\hat{f}(h)|^2)^{1/3}, \tag{11}$$

where

$$\hat{f}(h) = \int_0^1 \exp(2\pi i h x) df(x)$$

denotes the Fourier-Stiltjies transform of f.

We note that the well known LeVeque's inequality (see [3] or [2 : p. 111]) is a special case of (11). A result of Niederreiter [5] which improves a theorem of Elliott [1] and generalizes LeVeque's inequality is also a special case of (11).

Proof. Setting in Corollary 1 $\varphi(x)=x^2$ and applying it to the function

$$f(x) - \int_0^1 f(t)dt$$

we obtain

$$(1/12L)[f]^3 \leqq \int_0^1 f(x)^2 dx - \left(\int_0^1 f(x)dx\right)^2. \tag{12}$$

This completes the proof of (11) since the right-hand side of (12) is equal to

$$(1/2\pi^2) \textstyle\sum_{h=1}^{\infty} (1/h^2) |\hat{f}(h)|^2. \qquad \text{Q.E.D.}$$

We note that an inequality of Yurinskii [12] for the closeness of two distributions (mod 1) is a special case of (3). Another inequality of the type (11) which generalizes the well known Erdös-Turán inequality was given by Proinov in [9].

(to be continued.)

References

[1] P. D. T. A. Elliott: On distribution functions (mod 1). Quantitative Fourier inversion. J. Number Theory, 4, 509–522 (1972).
[2] L. Kuipers and H. Niederreiter: Uniform Distribution of Sequences. Wiley (1974).
[3] W. J. Le Veque: An inequality connected with Weyl's criterion for uniform distribution. Proc. Symp. Pure Math., vol. 8, Amer. Math. Soc., pp. 22–30 (1965).
[4] H. Niederreiter: Metric theorems on the distribution of sequences. ibid., vol. 24, Amer. Math. Soc., pp. 195–212 (1973).

[5] H. Niederreiter: Quantitative version of a result of Hecke in the theory of uniform distribution mod 1. Acta Arith., **28**, 321–339 (1975).

[6] ——: Résultats nouveaux dans la théorie quantitative de l'équirépartition. Lecture Notes in Math., vol. 475, Springer, pp. 132–154 (1975).

[7] P. D. Proinov: Generalization of two results on the theory of uniform distribution. Proc. Amer. Math. Soc., **95**, 527–532 (1985).

[8] ——: On an inequality in the theory of uniform distribution. C. R. Acad. Sci. Bulgare, **38**, 1465–1468 (1985).

[9] ——: On the Erdös-Turán inequality on uniform distribution. I, II. Proc. Japan Acad., **64A**, 27–28, 49–52 (1988).

[10] I. Schoenberg: Über die asymptotische Verteilung reeller Zahlen mod 1. Math. Z., **28**, 171–199 (1928).

[11] I. M. Sobol': Multidimensional quadrature formulae and Haar functions. Moscow (1969) (in Russian).

[12] V. V. Yurinskii: On inequalities for large deviations for certain statistics. Theor. Probability Appl., **16**, 385–387 (1971).