102. On an Extension of the James-Whitehead Theorem about Sphere Bundles over Spheres[†)]

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1. Statement of results. Let W be a handlebody obtained by gluing r, q-handles to a $(p+q+1)$-disk, and let $\mathscr{H}(p+q+1, r, q)$ be the set of such handlebodies. In this paper, I announce homotopy classification theorems of the boundaries of handlebodies of $\mathscr{H}(p+q+1, r, q)$ in the following two cases:

(1) $\quad (p, q)=(n-1, n+1) \quad (n\geq 4),$

(2) $\quad (p, q)=(n-2, n+1) \quad (n\geq 6).$

Such classifications are equivalent to those of simply connected closed m-manifolds M $(m=p+q)$ with $H_i(M)=0$ except for $i=0, p, q, m$ and with the tangent bundle which is trivial on its p-skeleton (this is satisfied if $p\equiv 3, 5, 6, 7 \bmod 8$). Henceforth, manifolds are connected, smooth, and oriented, and homotopy equivalences and diffeomorphisms are orientation preserving.

There exists an invariant system $(H; \phi, \alpha)$ which determines W up to diffeomorphism (cf. [4]). Here, $H=H_q(W)$, $\phi: H\times H\to Z_2=\pi_q(S^{p+1})$ is a symmetric bilinear form, and $\alpha: H\to\pi_{q-1}(SO_{p+1})$ is a quadratic form, which assigns, to each $x\in H\cong\pi_q(W)$, the characteristic element of the normal bundle of the imbedded q-sphere representing x. W is called of *type* 0 if $\phi=0$, of *type* II if $\phi(x, x)=0$ for any $x\in H$ and $\operatorname{rank}\phi=r$, and of *type* (0+II) if $\phi(x, x)=0$ for any $x\in H$ and $0<\operatorname{rank}\phi<r$. Note that ϕ is a homotopy invariant of ∂W by Proposition 1 of [2, II]. Our main purpose is to determine the necessary and sufficient condition for the boundaries of handlebodies to be homotopy equivalent using the invariant systems.

The following diagram is commutative up to sign:

$$\begin{array}{ccccc} & \overset{\partial}{\nearrow} & \pi_{q-1}(SO_p) & \xrightarrow{S} & \pi_{q-1}(SO_{p+1}) \\ \pi_q(S^p) & & \downarrow J & & \downarrow J \\ & \underset{[\ ,\iota_p]=P}{\searrow} & \pi_{m-1}(S^p) & \xrightarrow{E} & \pi_m(S^{p+1}), \quad m=p+q. \end{array}$$

Let $\lambda: S(\pi_{q-1}(SO_p))\to\pi_{m-1}(S^p)/\operatorname{Im}P$ be the homomorphism defined by $\lambda(S\xi)=\{J\xi\}$, which does not depend on the choice of ξ. Put $\theta=\eta_{n-1}$ if $(p, q)=(n-1, n+1)$ $(n\geq 4)$, and $\theta=\eta_{n-2}^2$ if $(p, q)=(n-2, n+1)$ $(n\geq 6)$. The inclusion map $i: S^p\to S^p\cup_\theta D^q$ induces the homomorphisms $i_*: \pi_{m-1}(S^p)\to\pi_{m-1}(S^p\cup_\theta D^q)$ and $\bar{i}_*: \pi_{m-1}(S^p)/\operatorname{Im}P\to\pi_{m-1}(S^p\cup_\theta D^q)/i_*(\operatorname{Im}P)$. We define $\bar{\lambda}: S(\pi_{q-1}(SO_p))\to\pi_{m-1}(S^p\cup_\theta D^q)/i_*(\operatorname{Im}P)$ by $\bar{\lambda}=\bar{i}_*\circ\lambda$.

Let W, W' be the handlebodies of $\mathscr{H}(p+q+1, r, q)$ with the invariant

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systems $(H;\phi,\alpha)$, $(H';\phi',\alpha')$ respectively, in one of the cases (1), (2). For $x\in H$, $\phi(x,x)=0$ implies that the normal p-sphere bundle of the imbedded q-sphere representing x admits a cross-section. So, $\alpha(x)$ belongs to $S(\pi_{q-1}(SO_p))\subset\pi_{q-1}(SO_{p+1})$.

Theorem 1. *Let W, W' be of type 0. Then, $\partial W, \partial W'$ are homotopy equivalent if and only if there exists an isomorphism $h: H\to H'$ such that $\lambda\circ\alpha=\lambda\circ(\alpha'\circ h)$.*

In the above theorem, $\partial W, \partial W'$ correspond to connected sums of r p-sphere bundles over q-spheres admitting cross-sections. So, it is known from Theorem 1 of [2, I] shown as an extension of the James-Whitehead Theorem [3]. We insert it to compare the following theorems with it.

Theorem 2. *Let W, W' be of type* II. *Then, $\partial W, \partial W'$ are homotopy equivalent if and only if there exists an isomorphism $h: H\to H'$ such that $\phi=\phi'\circ(h\times h)$ and $\bar\lambda\circ\alpha=\bar\lambda\circ(\alpha'\circ h)$.*

Theorem 3. *Let W, W' be of type* (0+II). *Then, $\partial W, \partial W'$ are homotopy equivalent if and only if there exists an isomorphism $h: H\to H'$ such that $\phi=\phi'\circ(h\times h)$, $\bar\lambda\circ\alpha=\bar\lambda\circ(\alpha'\circ h)$ and furthermore there exists a direct sum decomposition $H=H_0+H_1$ orthogonal with respect to ϕ such that $\phi|H_0\times H_0=0$, $\phi|H_1\times H_1$ is non-singular, and $\lambda\circ\alpha=\lambda\circ(\alpha'\circ h)$ on H_0.*

If W is of type II or of type (0+II), then ∂W can never be represented as a connected sum of p-sphere bundles over q-spheres even up to homotopy equivalence (cf. Lemma 1.1 of [1]). In case that there exists an element $x\in H$ such that $\phi(x,x)\neq 0$, W is called of *type* I if $\operatorname{rank}\phi=r$, of *type* (0+I) if $0<\operatorname{rank}\phi<r$. Those boundaries of such handlebodies are represented as connected sums of p-sphere bundles over q-spheres, and the homotopy classification is completed in [2, III].

2. Outline of the proofs. The detailed proofs of the above theorems will appear elsewhere. Here, I outline the proof for Theorem 2 since Theorem 3 is obtained similarly.

Every $W\in\mathcal{H}(p+q+1,r,q)$ can be represented as $W=D^{m+1}\cup_{\{f_i\}}\{\cup_{i=1}^r D_i^q \times D_i^{p+1}\}$ $(m=p+q)$ so that $D_i^q\times o$, $i=1,2,\cdots,r$, represent the given basis $e_1,e_2,\cdots,e_r$ of $H\cong H_q(W,D^{m+1})$. Let W be of type II. Then, there exists a basis $e_1,e_2,\cdots,e_r$ $(r=2s)$ symplectic w.r.t. ϕ (cf. Lemma 1.1 of [1]). So, W can be represented by using it. Let $K_\phi=\vee_{i=1}^s\{(S_{2i-1}^p\cup_\theta D_{2i}^q)\vee(S_{2i}^p\cup_\theta D_{2i-1}^q)\}$. We denote the orientation generator of $\pi_p(S_i^p)$ by ι_p^i and similarly $(D_j^q)\in \pi_q(K_\phi,\vee_{i=1}^r S_i^p)$ by σ_q^j.

Lemma 4. *The boundary of W of type* II *has a cellular decomposition $\partial W\simeq K_\phi\cup_\omega D^m$, where ω is given by $\omega=\mu+i_*(\iota_p^1\circ J\beta_1+\iota_p^2\circ J\beta_2+\cdots+\iota_p^r\circ J\beta_r)$. Here, $S(\beta_i)=\alpha(e_i)$, $\beta_i\in\pi_{q-1}(SO_p)$, $i=1,2,\cdots,r$, and $i_*:\pi_{m-1}(\vee_{i=1}^r S_i^p)\to \pi_{m-1}(K_\phi)$ is induced from the inclusion map. $\mu\in\pi_{m-1}(K_\phi)$ is of infinite order and corresponds to $[\sigma_q^1,\iota_p^1]+\cdots+[\sigma_q^r,\iota_p^r]$ under $j_*:\pi_{m-1}(K_\phi)\to\pi_{m-1}(K_\phi,\vee_{i=1}^r S_i^p)$. μ does not depend on α and is called "fundamental homotopy class".*

Proof (Sketch). We may assume that $r=2$ w.l.o.g. Since $S_i^p=D_i^p/\partial D_i^p$, each handle $D_i^p\times S_i^p$ of ∂W can be considered as $D_i^q\times D_i^p=D_i^m$ attached to

$D_i^q \times y_i$ ($y_i \in S_i^p$). So, by connecting $f_1(S_1^{q-1}\times D_1^{p+1})$, $f_2(S_2^{q-1}\times D_2^{p+1})$ with a thin band in S^m, we have $\partial W=\tilde{Y}\cup(D_1^q\cup D_2^q)\cup(D_1^m \natural D_2^m)$, where $\tilde{Y}=S^m-\operatorname{Int}\{f_1(S_1^{q-1}\times D_1^{p+1})\natural f_2(S_2^{q-1}\times D_2^{p+1})\}$ and $\natural$ denotes the boundary connected sum. Let ω_i be the attaching map of D_i^m. $D_i^q=D_i^q\times y_i$ can be taken as a half of the cross-section by β_i of the normal p-sphere bundle for e_i. ω_i is determined by the situation of a thin neighbourhood of D_i^q in the handle $D_i^q\times S_i^p\subset\partial W$. The attaching map ω of $D^m=D_1^m\natural D_2^m$ is given by $\omega_1\,\#\,\omega_2$. Let Z be $f_1(S_1^{q-1}\times S_1^p)\,\#\,f_2(S_2^{q-1}\times S_2^p)$ with D_1^q, D_2^q attached. Z is included in $\tilde{Y}\cup(D_1^q\cup D_2^q)$ and we can see that Z is homotopy equivalent to $S^{m-1}\vee S_1^p\vee S_2^p$. Then, it is checked that $\omega_1\,\#\,\omega_2\colon \partial D^m\to Z$ corresponds to $\iota_{m-1}+\iota_p^1\circ J\beta_1+\iota_p^2\circ J\beta_2$ under the homotopy equivalence. $\tilde{Y}$ is deformed to $S_1^p\vee S_2^p$ and the attaching maps of D_i^q, $i=1,2$, correspond to the linking elements of the link $f_1(S_1^{q-1}\times o)\cup f_2(S_2^{q-1}\times o)\subset S^m$ which can be evaluated by ϕ. So, the retraction extends to a homotopy equivalence $\tilde{Y}\cup D_1^q\cup D_2^q\simeq K_\phi$. The subspace Z is also mapped to K_ϕ. Thus, we have $\omega=\mu+i_*(\iota_p^1\circ J\beta_1+\iota_p^2\circ J\beta_2)$, where μ is of infinite order since we know that $j_*(\mu)=[\sigma_q^1, \iota_p^1]+[\sigma_q^2, \iota_p^2]$ after a calculation.

Sufficiency proof for Theorem 2. Take the symplectic basis $e_i'=h(e_i)$, $i=1,2,\cdots,r$, for H'. We have $\partial W\simeq K_\phi\cup_\omega D^m$, $\partial W'\simeq K_\phi\cup_{\omega'}D^m$. Let $\bar{\iota}_p^i\colon S^p\cup_\theta D^q\to K_\phi$ be a canonical extension of $\iota_p^i\colon S^p\to S_i^p$. Then, $\omega=\mu+(\bar{\iota}_p^1\circ i_*J\beta_1+\cdots+\bar{\iota}_p^r\circ i_*J\beta_r)$ for $i_*\colon\pi_{m-1}(S^p)\to\pi_{m-1}(S^p\cup_\theta D^q)$, and similarly for ω'. Since $\bar{\lambda}(S\beta_i)=\bar{\lambda}\alpha(e_i)=\bar{\lambda}\alpha'(e_i')=\bar{\lambda}(S\beta_i')$ from the condition, we have $J\beta_i-J(\beta_i'-\partial\gamma_i)=\delta_i$ for some $\gamma_i\in\pi_q(S^p)$, $\delta_i\in\operatorname{Ker} i_*$. Put $\beta_i''=\beta_i'-\partial\gamma_i$, $i=1,2,\cdots,r$. Since $S\beta_i''=S\beta_i'=\alpha'(e_i')$, we can take another decomposition $\partial W'\simeq K_\phi\cup_{\omega''}D^m$ with $\omega''=\mu+(\bar{\iota}_p^1\circ i_*J\beta_1''+\cdots+\bar{\iota}_p^r\circ i_*J\beta_r'')$. So, $\omega''\simeq\omega$ and therefore $\partial W\simeq\partial W'$.

Lemma 5. *Let W, W' be of type* II *and let $f\colon\partial W\to\partial W'$ be a homotopy equivalence. Then, there exist cellular decompositions $\partial W\simeq K_\phi\cup_\omega D^m$, $\partial W'\simeq K_{\phi'}\cup_{\omega'}D'^m$ such that $f_*\mu-\mu'=i_*'([\theta_1', \iota_p'^1]+\cdots+[\theta_r', \iota_p'^r])$ for certain elements $\theta_i'\in\pi_q(\vee_{j=1}^r S_j'^p)$, $i=1,2,\cdots,r$, where μ, μ' are fundamental homotopy classes.*

Proof (Sketch). $K_{\phi'}$ is a copy of K_ϕ. By Lemma 2.1 of [2, II], we can take such decomposition that $f_*(\iota_p^i)=\iota_p'^i$, $i=1,2,\cdots,r$, (hence $f\,|\vee_{i=1}^r S_i^p=\mathbf{1}$) and $\bar{f}_*(\sigma_q^j)=\sigma_q'^j$, $j=1,2,\cdots,r$, where $\bar{f}\colon(K_\phi, \vee_{i=1}^r S_i^p)\to(K_{\phi'}, \vee_{i=1}^r S_i'^p)$. Let $K=K_\phi\cup K_{\phi'}$ with $S_i^p, S_i'^p$ identified for $i=1,2,\cdots,r$. Let $m\colon K_\phi\to K$, $m'\colon K_{\phi'}\to K$ be inclusion maps, and put $\bar{\mu}=m_*\mu$, $\bar{\mu}'=m_*'\mu'$. Let $A=\vee_{i=1}^r(S_i^q\vee S_i^p)\cup_\chi D^m$, where $\chi=[\iota_q^1, \iota_p^1]+\cdots+[\iota_q^r, \iota_p^r]$. Then, by a certain geometric construction, there exists a map $g\colon A\to K\cup_{\bar{\mu}}D^m\cup_{\bar{\mu}'}D'^m$ such that $\bar{\mu}-\bar{\mu}'=g_*([\iota_q^1, \iota_p^1]+\cdots+[\iota_q^r, \iota_p^r])$. Let $r'\colon K\to K_{\phi'}$ be the retraction defined by $r'\,|K_\phi=f$ and $r'\,|K_{\phi'}=\mathbf{1}$. Then, $f_*(\mu)-\mu'=r_*'(\bar{\mu}-\bar{\mu}')=(r'\circ g)_*([\iota_q^1, \iota_p^1]+\cdots+[\iota_q^r, \iota_p^r])$. From the construction of g, we know that $(r'\circ g)_*\iota_p^i=i_*'\iota_p'^i$, $l_*(g_*\iota_q^i)=\overline{m}_*\sigma_q^i-\overline{m}_*'\sigma_q'^i$, where $l\colon K\to(K, \vee_{i=1}^r S_i^p)$ is the inclusion map and $\overline{m}, \overline{m}'$ are the relativizations of m, m'. Hence, for $j_*'\colon\pi_q(K_{\phi'})\to\pi_q(K_{\phi'}, \vee_{i=1}^r S_i'^p)$, we have $j_*'(r'\circ g)_*\iota_q^i=\bar{r}_*'l_*(g_*\iota_q^i)=\bar{r}_*'(\overline{m}_*\sigma_q^i-\overline{m}_*'\sigma_q'^i)=\bar{f}_*\sigma_q^i-\sigma_q'^i=0$. Thus, there exists $\theta_i'\in\pi_q(\vee_{j=1}^r S_j'^p)$ such that $i_*'(\theta_i')=(r'\circ g)_*\iota_q^i$. Therefore, $f_*(\mu)-\mu'=i_*'([\theta_1', \iota_p'^1]+\cdots+[\theta_r', \iota_p'^r])$.

Necessity proof for Theorem 2. Since $f_*(\omega)=\omega'$, we have $f_*(\mu)-\mu'+i'_*\{\iota'^1_p\circ(J\beta_1-J\beta'_1)+\cdots+\iota'^r_p\circ(J\beta_r-J\beta'_r)\}=0$. In Lemma 5, $\theta'_i=\Sigma^r_{j=1}\iota'^j_p\circ\theta_{ij}+\Sigma_{j<k}[\iota'^j_p,\iota'^k_p]\circ\theta_{ijk}$ for certain $\theta_{ij}\in\pi_q(S^p)$, $\theta_{ijk}\in\pi_q(S^{2p-1})$, $j,k=1,2,\cdots,r$. So, we have $\Sigma_i i'_*a_i+\Sigma_{i<j}i'_*b_{ij}+\Sigma_{i\geqq j<k}i'_*c_{ijk}=0$, where $a_i=\iota'^i_p\circ(J\beta_i-J\beta'_i+[\theta_{ii},\iota_p])$ and those consisting of the basic products of weight 2 (weight 3) are included in the second (third) term. Hence, by the argument in Assertion 3 of [2, II, p. 321], we have $\Sigma^r_{i=1}i'_*a_i=0$, and therefore, $i'_*a_i=0$, $i=1,2,\cdots,r$. Define $\bar\iota'^i_p$ similarly to $\bar\iota^i_p$. Since $i'\circ\iota'^i_p=\bar\iota'^i_p\circ i$ and $(\bar\iota'^i_p)_*$ is injective, we have $i_*(J\beta_i-J\beta'_i+[\theta_{ii},\iota_p])=0$, and so $i_*(J\beta'_i)-i_*(J\beta_i)=i_*(P\theta_{ii})$. This implies $\bar\lambda\alpha(e_i)=\bar\lambda\alpha'(e'_i)$, $i=1,2,\cdots,r$. Then, an isomorphism $h: H\to H'$ defined by $h(e_i)=e'_i$, $i=1,2,\cdots,r$, will satisfy the conditions.

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