101. *A Construction of Negatively Curved Manifolds*

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§1. **Introduction.** Let V be a complete Riemannian manifold with $-b<K<-a<0$ and $\mathrm{vol}(V)<\infty$. Then it is known that each end of V is an infranilmanifold ([1], [2]).

But if we change the condition $-b<K<-a<0$ to $-b<K<0$, then the conclusion does not hold in general. In this paper we will give a counter-example; if the dimension is bigger than three, there is a complete manifold V with $-b<K<0$ and $\mathrm{vol}(V)<\infty$ such that the end is not an infranilmanifold, and in the case that the dimension is three, the end is a torus.

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§2. **Theorem and its proof. Theorem.** *Let V be a closed manifold with $K\equiv -1$ and W a closed totally geodesic submanifold of codimension 2 in V.*

Then $V\setminus W$ admits a complete metric with $-a<K<0$ and $\mathrm{vol}(V\setminus W)<\infty$, where $a>0$.

Remark 1. A pair (V, W) with the above property exists.

Remark 2. In this theorem, the end of $V\setminus W$ is a S^1-bundle over a hyperbolic manifold W, which is not an infranilmanifold.

Proof. Let $\sigma=\mathrm{inj}(W;V)$, and take a σ-neighborhood U of W in V. We introduce a polar coordinate (w,θ,r) on U. Then $U=W\times S^1\times(0,\sigma)$ and we can write the hyperbolic metric g_V of V as follows on U ([4], [3]),

(1) $$g_V=\cosh^2(r)g_W+\sinh^2(r)d\theta^2+dr^2 \qquad (0\le\theta\le 2\pi,\ 0\le r\le\sigma)$$

where g_W denotes the induced metric on W.

We are going to change the metric g_V to a new metric $h_{V'}$ on $V'=V\setminus W$ as follows. Using a positive function $f(r)$, we set

(2) $$h_{V'}=\cosh^2(r)g_W+\sinh^2(r)d\theta^2+f^2(r)dr^2 \qquad (0\le\theta\le 2\pi,\ 0\le r\le\sigma).$$

To choose a suitable function $f(r)$, we compute the sectional curvature K_h of the metric $h_{V'}$. First, note that a vector field ξ on W naturally extends to a vector field on U, and we also denote it by ξ. The Riemannian connection ∇ of $h_{V'}=\langle\ ,\ \rangle$ is given as follows, where D denotes the Riemannian connection on W, and $\xi,\zeta,\cdots$ denote vector fields on W or their extentions to U.

$$\begin{cases}\nabla_\xi\zeta=D_\xi\zeta-\tanh(r)\langle\xi,\zeta\rangle\dfrac{\partial}{\partial r}\\[2mm] \nabla_\xi\dfrac{\partial}{\partial\theta}=\nabla_{\partial/\partial\theta}\xi=0\end{cases}$$

$$(3)\qquad \begin{cases} \nabla_{\xi}\dfrac{\partial}{\partial r}=\nabla_{\partial/\partial r}\xi=\tanh(r)\xi \\ \nabla_{\partial/\partial\theta}\dfrac{\partial}{\partial r}=\nabla_{\partial/\partial r}\dfrac{\partial}{\partial\theta}=\coth(r)\dfrac{\partial}{\partial\theta} \\ \nabla_{\partial/\partial\theta}\dfrac{\partial}{\partial\theta}=-\sinh(r)\cosh(r)\dfrac{\partial}{\partial r} \\ \nabla_{\partial/\partial r}\dfrac{\partial}{\partial r}=\dfrac{f'(r)}{f(r)}\dfrac{\partial}{\partial r} \end{cases}$$

Thus, the curvature tensor R of $h_{V'}$ is given as follows,

$$(4)\qquad \begin{cases} R\left(\dfrac{\partial}{\partial r},\ \dfrac{\partial}{\partial\theta}\right)\dfrac{\partial}{\partial r}=\left(1-\dfrac{f'(r)}{f(r)}\coth(r)\right)\dfrac{\partial}{\partial\theta} \\ R\left(\dfrac{\partial}{\partial r},\ \dfrac{\partial}{\partial\theta}\right)\dfrac{\partial}{\partial\theta}=-\sinh(r)\left(\sinh(r)+\cosh(r)\dfrac{f'(r)}{f(r)}\right)\dfrac{\partial}{\partial r} \\ R\left(\dfrac{\partial}{\partial r},\ \dfrac{\partial}{\partial\theta}\right)\xi=0 \\ R\left(\dfrac{\partial}{\partial r},\ \xi\right)\dfrac{\partial}{\partial r}=\left(1-\dfrac{f'(r)}{f(r)}\tanh(r)\right)\xi \\ R\left(\dfrac{\partial}{\partial r},\ \xi\right)\zeta=-\left(1+\dfrac{f'(r)}{f(r)}\tanh(r)\right)\langle\xi,\zeta\rangle\dfrac{\partial}{\partial r} \\ R\left(\dfrac{\partial}{\partial r},\ \xi\right)\dfrac{\partial}{\partial\theta}=0 \\ R\left(\dfrac{\partial}{\partial\theta},\ \xi\right)\dfrac{\partial}{\partial r}=0 \\ R\left(\dfrac{\partial}{\partial\theta},\ \xi\right)\dfrac{\partial}{\partial\theta}=\sinh^2(r)\xi \\ R\left(\dfrac{\partial}{\partial\theta},\ \xi\right)\zeta=-\langle\xi,\zeta\rangle\dfrac{\partial}{\partial\theta} \\ R(\xi_1,\xi_2)\zeta=\langle\xi_1,\zeta\rangle\xi_2-\langle\xi_2,\zeta\rangle\xi_1 \\ R(\xi_1,\xi_2)\dfrac{\partial}{\partial r}=0 \\ R(\xi_1,\xi_2)\dfrac{\partial}{\partial\theta}=0. \end{cases}$$

Then it follows that for the curvature K of $h_{V'}$,

$$(5)\qquad \begin{cases} K(\xi_1\wedge\xi_2)=-1 \\ K\left(\xi\wedge\dfrac{\partial}{\partial\theta}\right)=-1 \\ K\left(\xi\wedge\dfrac{\partial}{\partial r}\right)=\dfrac{-1+\dfrac{f'(r)}{f(r)}\tanh(r)}{f^2} \\ K\left(\dfrac{\partial}{\partial\theta}\wedge\dfrac{\partial}{\partial r}\right)=\dfrac{-1+\dfrac{f'(r)}{f(r)}\coth(r)}{f^2}. \end{cases}$$

Furthermore, by a easy computation, the curvature at every 2-plane is a convex combination of the above numbers -1,

$$\frac{-1+\frac{f'(r)}{f(r)}\tanh(r)}{f^2}, \quad \text{and} \quad \frac{-1+\frac{f'(r)}{f(r)}\coth(r)}{f^2}.$$

Here we need a following lemma.

Lemma. *There is a function*

$$f:(0,\sigma)\longrightarrow \boldsymbol{R}_+$$

such that

(6.1) $f(r)=1 \quad \left(\frac{\sigma}{2}\leq r\leq\sigma\right)$

(6.2) $f'(r)\leq 0$

(6.3) $\int_0^\sigma f(r)dr=\infty$

(6.4) $\int_0^\sigma f(r)\sinh(r)dr<\infty$

(6.5) $\left|\frac{f'(r)}{f^3(r)}\coth(r)\right|$ *is bounded,* $(0<r\leq\sigma)$

(6.6) $\left|\frac{f'(r)}{f^3(r)}\tanh(r)\right|$ *is bounded,* $(0<r\leq\sigma)$.

Proof of lemma. At first, (6.6) follows from (6.5). Define a function ϕ as follows,

$$\phi(r)=\frac{1}{\sqrt{r}\sinh(r)} \qquad (0<r\leq\sigma).$$

Then we have

(7.1) $\int_0^\sigma \phi(r)dr=\infty$

(7.2) $\int_0^\sigma \phi(r)\sinh(r)dr<\infty$

(7.3) $\left|\frac{\phi'(r)}{\phi^3(r)}\coth(r)\right|$ is bounded, $(0<r\leq\sigma)$.

Here we may assume $\phi(\sigma/4)>1$, taking σ small. Then it is easy to choose a function $f(r)$ $(0<r\leq\sigma)$ such that

(8.1) $f(r)=1 \quad \left(\frac{\sigma}{2}\leq r\leq\sigma\right)$

(8.2) $f(r)=\phi(r) \quad \left(0<r<\frac{\sigma}{4}\right)$

(8.3) $f'(r)\leq 0$.

From (7.1)–(7.3) and (8.1)–(8.3), it follows that $f(r)$ is a required function. Hence lemma is shown.

Using $f(r)$ in lemma, we define a new metric $h_{U'}$ on $U'=U\backslash W$ as follows,

(9) $h_{U'}=\cosh^2(r)g_W+\sinh^2(r)d\theta^2+f^2(r)dr^2.$

By (6.1), we can extend $h_{U'}$ to a metric $h_{V'}$ on V' by letting $h_{V'}=g_V$ on $V\backslash U$. Then (6.2), (6.5), (6.6), and (5) imply that the curvature K_h of $h_{V'}$ satisfies $-a<K_h<0$ for some $a>0$. Further, the completeness of $h_{V'}$ follows from

(6.3), and (6.4) implies $\mathrm{vol}_h(V')<\infty$. Hence $h_{V'}$ is a required metric on V', and theorem is proved.

References

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