100. A Cohomological Construction of Swan Representation over the Witt Ring. II

By Osamu HYODO
Department of Mathematics, Nara Women’s University
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This is continued from [0].

3. In this section we give the construction of Swan representations. Let $K=k((T^{-1}))$ be a complete discrete valuation field and $M$ be a finite Galois extension of $K$ with Galois group $G$. N. M. Katz proved the following Theorem ([7] Theorem (1.4.1)). There exists a canonical finite etale Galois covering

$$U \longrightarrow G_{m,k}=\text{Spec } k[T, T^{-1}]$$

which satisfies the following properties.

1. $U \otimes_{k[T, T^{-1}]} \text{Spec } k((T^{-1})) \cong \text{Spec } M$
2. $U \otimes_{k[T, T^{-1}]} \text{Spec } k((T))$ is a disjoint union of the spectra of tamely ramified extensions of $k((T))$.

We denote by $X=g \mapsto P^i$ the compactification of $U \longrightarrow G_{m,k}$. Note that $g$ factors as $X \longrightarrow P^i \longrightarrow P^i_m$, where $m$ denotes the residue field of $M$. We denote by $D_0$ (resp. $D_\infty$) the inverse image of $T=0$ (resp. $T=\infty$) with reduced scheme structure. Then $X \setminus U=D_0 \amalg D_\infty$. Let $\Omega_X^i (\log D_0-\log D_\infty)$ be the de Rham-Witt complex with logarithmic poles along $D_0$ and with minus logarithmic poles along $D_\infty$ as in §1. As $D_0$ and $D_\infty$ are stable under the action of $G$, $\sigma \in G=\text{Gal } (U/G_{m,k})$ acts on the free $W$-module $H^i(X, \Omega_X^i (\log D_0-\log D_\infty))$ by transportation of structures. The following Proposition shows that this is the desired space of the Swan representation of $G$.

Proposition. The trace of the action of $\sigma \in G$ on $H^i(X, \Omega_X^i (\log D_0-\log D_\infty))$ coincides with $Sw_\infty(\sigma)$.

In the following we denote the alternating sum of the trace of the action of $\sigma$ on free $W(k)$-modules by

$$\text{Tr}(\sigma) = R\Gamma(X, -):=\sum_{q \geq 0} (-1)^q \text{Tr}(\sigma; H^q(X, -)).$$

By Lemma and exact sequences (***) in §1, it suffices to show

$$\text{Tr}(\sigma; R\Gamma(X, \Omega_X^i)) = \begin{cases} d^i + (-Sw_0(\sigma) + f) & \text{for } \sigma \in I, \\ 0 & \text{for } \sigma \in I, \end{cases}$$

where $d^i$ denotes the degree of the closed subscheme of $D_i$ fixed by $\sigma$ and $f=[m: k]$ coincides with degree of $D_\infty$ over $k$.

The proof of this formula is the same as the proof of the Weil formula [4] §5: The case $\sigma=1$ is the Hurwitz formula. The case $\sigma \neq 1$ is deduced
from the fixed point formula (crystalline cohomology is a Weil cohomology theory [3]). We omit the detail.

Remark and question. (1) Contrary to the $l$-adic case, $Sw_{G,p}$ can not always be realized as a projective $W(k)[G]$-module. This phenomenon seems to suggest that one can not expect the "Grothendieck-Ogg-Shafarevich formula" for crystals defined over open smooth curves. (cf. [7] § (1.6).)

(2) Nevertheless, are there nice theory of the Swan conductor (or irregularity) for crystals?

Reference*)


*) [1]–[8] as in [0].