99. *Tables of Ideal Class Groups of Real Quadratic Fields*

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§1. **Introduction.** A table of ideal class groups of imaginary quadratic fields $Q(\sqrt{-m})$ was given in [5] for $m<100,000$. In this note we shall give corresponding tables for ideal class groups in narrow sense and in wide sense for real quadratic fields $Q(\sqrt{m})$ for $m<100,000$. As in [5], we use the expression $(a, b, \cdots, c)$ to denote the type of finite abelian group which is the direct product of cyclic groups of order $a, b, \cdots, c$, $aZ\subset bZ\subset \cdots\subset cZ$. The ideal class groups in wide and narrow sense, the class numbers in wide and narrow sense, the fundamental unit, the 2-ranks of the ideal class groups in wide and narrow sense of $Q(\sqrt{m})$ and the number of rational primes ramified in $Q(\sqrt{m})$ are denoted by $C(m)$, $C'(m)$, $h(m)$, $h'(m)$, $\varepsilon(m)$, $r(m)$, $r'(m)$ and $t(m)$, (sometimes simply by C, C', h, h', ε, r, r', t) respectively. It is well known that $h'=h$ or $2h$ accoring as $N\varepsilon=-1$ or $+1$ and $r'=t-1$. We recall that a table of $h(m)$ and $N\varepsilon(m)$ is given in [4] for $m<100,000$. The method of our calculation is based on [2] Chapter 5. It was done by micro VAX-II and the computer time for making these tables was about 40 hours.

§2. **Ideal class groups in narrow sense.** Our Table I gives the types $(a, b, \cdots, c)$ of $C'(m)$ for all $m<100,000$ except in the following two cases:

(1) $C'(m)$ is cyclic.

(2) $C'(m)$ is of the type $(2a', 2, \cdots, 2)$ and $t>2$.

Thus when m is not found in Table I, and $t=1$ or 2, then $C'(m)$ is cyclic, and when $t>2$ then $C'(m)$ is of the type $(2a', 2, \cdots, 2)$ with $a'=h'/2^{t-1}$.

§3. **Ideal class groups in wide sense.** If $N\varepsilon(m)=-1$, it is well-known that $C'(m)$ and $C(m)$ are of the same type. We have furthermore the following theorem.

Theorem. *Let $R(m)$ be the set of rational primes ramified in $Q(\sqrt{m})$ (i.e. the set of prime divisors of the discriminant of $Q(\sqrt{m})$).*

(1) *If $R(m)$ contains a prime $\equiv 3 \pmod 4$, then*

$$r(m)=r'(m)-1=t-2.$$

(2) *Otherwise* $r(m)=r'(m)=t-1$.

The proof of this theorem is implicitly contained in [1] or in [3], but this explicit formulation was communicated to us by Prof. Iwasawa. We add here a short proof for convenience.

Proof. In case (1), the norm of the fundamental unit is 1 and there is no number $\theta\in Q(\sqrt{m})$ satisfying $N(\theta)=-1$. So $r=t-2$ ([3] p. 257).

In case (2), we can conclude from calculation of the Hilbert Symbol that there exists a number $\theta \in \boldsymbol{Q}(\sqrt{m})$ with $N(\theta)=-1$ by theorem of Minkowski-Hasse. Q.E.D.

In case (1) above, $C(m)$ is obtained from $C'(m)$ as follows. Let $(2^{\alpha}, 2^{\beta}, \cdots, 2^{\gamma}, 2^{\delta})$ be the 2-Sylow group of C'. Then δ should be 1 and the 2-Sylow group of C is $(2^{\alpha}, 2^{\beta}, \cdots, 2^{\gamma})$. This case is therefore not contained in Table II.

Also in case (2), m is not contained in Table II if $N\varepsilon(m)=-1$. Now consider the case (2) with $N\varepsilon(m)=+1$. Let $(2^{\alpha}, 2^{\beta}, \cdots, 2^{\gamma}, 2, \cdots, 2)$ be the 2-Sylow group of $C'(m)$. In calculating the type of $C(m)$, we have found that in most cases, $C(m)$ is of the type $(2^{\alpha}, 2^{\beta}, \cdots, 2^{\gamma-1}, 2, \cdots, 2)$. These cases are also omitted from Table II, which gives all other 14 cases.

Table I

m	narrow sense	m	narrow sense	m	narrow sense
3026	(4, 4)	32402	(4, 4)	51610	(4, 4, 2)
4658	(4, 4)	32882	(4, 4)	51694	(6, 3)
4810	(4, 4, 2)	33218	(4, 4)	52658	(4, 4)
5986	(4, 4)	33245	(4, 4)	52882	(8, 4)
8738	(4, 4)	33538	(8, 4)	53507	(6, 3)
9266	(4, 4)	34697	(4, 4)	53678	(6, 3)
9554	(4, 4)	34945	(8, 4)	54466	(4, 4)
11713	(4, 4)	35122	(4, 4)	55145	(4, 4)
12002	(4, 4)	35506	(4, 4)	56163	(4, 4, 2)
12505	(8, 4)	36818	(4, 4)	56338	(4, 4)
12994	(12, 4)	37298	(4, 4)	56797	(4, 4)
13906	(4, 4)	37442	(8, 4)	58082	(4, 4)
14162	(4, 4)	37522	(4, 4)	58145	(4, 4)
14722	(8, 4)	38090	(4, 4, 2)	58466	(4, 4, 2)
15538	(8, 4)	40745	(4, 4)	58514	(4, 4)
15805	(4, 4)	41210	(4, 4, 2)	58546	(8, 4)
17266	(8, 4)	41474	(4, 4)	59602	(4, 4)
19346	(4, 4)	42466	(4, 4)	60418	(4, 4)
19618	(4, 4)	42486	(4, 4, 2)	60722	(4, 4)
19762	(8, 4)	42817	(6, 3)	61234	(4, 4)
19981	(4, 4)	43063	(6, 3)	62146	(8, 4, 2)
20002	(4, 4)	43486	(12, 6)	62402	(4, 4)
20162	(4, 4)	44251	(4, 4, 2)	62501	(3, 3)
21243	(4, 4, 2)	45445	(4, 4)	62687	(6, 3)
21605	(4, 4)	45746	(4, 4)	63058	(4, 4)
21922	(4, 4)	46274	(4, 4)	64226	(8, 4)
22321	(4, 4)	46658	(4. 4)	65042	(8, 4)
23659	(6, 6)	46754	(4, 4)	65221	(8, 4)
25610	(4, 4, 2)	47906	(8, 4)	66722	(4, 4)
25874	(4, 4)	48505	(4, 4)	67762	(4, 4)
26146	(8, 4)	49042	(4, 4, 2)	68237	(4, 4)
26245	(4, 4)	49405	(4, 4)	68482	(8, 4)
27634	(4, 4)	50354	(8, 4)	68839	(4, 4, 2)
28946	(4, 4)	50594	(4, 4)	69845	(4, 4)
31858	(12, 4)	50605	(8, 4)	70131	(4, 4, 2)
32009	(3, 3)	51538	(4, 4)	70274	(8, 4)

Table I (continued)

m	narrow sense	m	narrow sense	m	narrow sense
70418	(4, 4)	81362	(4, 4)	91083	(4, 4, 2)
70738	(4, 4)	82045	(4, 4)	91405	(8, 4)
71026	(4, 4)	82205	(4, 4)	91690	(4, 4, 2)
71111	(8, 4, 2)	82433	(8, 4)	92045	(4, 4)
71378	(8, 4)	82738	(4, 4)	92338	(12, 4)
71545	(4, 4)	83210	(4, 4, 2)	92786	(8, 4)
72242	(8, 4)	83414	(6, 6)	93026	(8, 4)
72329	(6, 3)	83762	(4, 4, 2)	93586	(16, 4)
72386	(4, 4)	83845	(4, 4)	93602	(8, 4)
72410	(4, 4, 2)	84082	(4, 4)	94505	(12, 4)
72802	(4, 4)	84706	(8, 4)	94546	(40, 4)
73474	(12, 4)	85431	(6, 6)	94963	(4, 4, 2)
73505	(8, 4)	85666	(6, 6, 2)	95234	(8, 4)
73805	(4, 4)	85762	(8, 4)	95290	(4, 4, 2)
75026	(4, 4, 2)	85969	(8, 4)	95693	(4, 4)
75545	(4, 4)	86578	(8, 4)	95845	(4, 4)
76162	(8, 4)	87106	(8, 4)	96889	(4, 4)
76994	(8, 4)	87841	(4, 4)	97138	(12, 4)
77857	(4, 4)	88162	(4, 4)	97719	(6, 6)
78098	(4, 4)	88502	(4, 4, 2)	97826	(12, 4)
78146	(8, 4)	88706	(8, 4)	98258	(4, 4)
79690	(4, 4, 2)	88978	(4, 4)	98810	(4, 4, 2)
79745	(4, 4)	89522	(8, 4)	99202	(8, 4)
79778	(8, 4)	89609	(4, 4)	99231	(4, 4, 2, 2)
79922	(4, 4)	89954	(4, 4)		
80189	(4, 4)	90626	(4, 4)		

Table II

m	narrow sense	wide sense	m	narrow sense	wide sense
14722	(8, 4)	(4, 4)	73505	(8, 4)	(4, 4)
37422	(8, 4)	(4, 4)	82433	(8, 4)	(4, 4)
50354	(8, 4)	(4, 4)	85969	(8, 4)	(4, 4)
58546	(8, 4)	(4, 4)	87106	(8, 4)	(4, 4)
65042	(8, 4)	(4, 4)	93586	(16, 4)	(8, 4)
71378	(8, 4)	(4, 4)	93602	(8, 4)	(4, 4)
72242	(8, 4)	(4, 4)	94546	(40, 4)	(20, 4)

References

[1] David Hilbert: Zahlbericht, § 77, Satz 108.

[2] Teiji Takagi: Lecture on the Elementary Theory of Numbers. Kyoritsu, 2nd ed. (1971) (in Japanese).

[3] ——: Algebraic Theory of Numbers. Iwanami, 2nd ed. (1971) (in Japanese).

[4] Hideo Wada: A table of ideal class numbers of real quadratic fields. Sophia Kokyuroku in Mathematics, no. 10 (1981).

[5] Hideo Wada and Michiyo Saito: A table of ideal class groups of imaginary quadratic fields. ibid., no. 28 (1988).