77. *On Pathwise Projective Invariance of Brownian Motion.* II

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In part I, we have obtained a measure preserving group action which is isomorphic to $SL(2, \boldsymbol{R})$ in the path space of Brownian motion $\{B(t;\omega)\}$;

$$(1)\qquad B^g(t;\omega)=(ct+d)B\left(\frac{at+b}{ct+d};\omega\right)-ctB\left(\frac{a}{c};\omega\right)-dB\left(\frac{b}{d};\omega\right),$$
$$g=\begin{pmatrix} a, & b \\ c, & d\end{pmatrix}\in SL(2,\boldsymbol{R}).$$

From this action we have deduced a symmetric property called P. Lévy's *projective invariance of Brownian motion.*

In this part we, using the terms of theory of unitary representation, determine the class of the above action a *discrete series representation* of index 2.

§ 4. Stochastic integral of Wiener type. Let $\{X(t;\omega);t\in\boldsymbol{R}\}$ be a Gaussian process with continuous path and $\varphi\in\mathcal{D}(\boldsymbol{R})$ be a test function. Define an integral of Wiener type;

$$(6)\qquad I(\varphi;\omega)\equiv-\int_{\boldsymbol{R}}\varphi'(t)X(t;\omega)dt.$$

An inner product is defined by

$$(7)\qquad (\varphi,\psi)\equiv\boldsymbol{E}[I(\varphi;\omega)\overline{I(\psi;\omega)}]=\iint\varphi'(t)\overline{\psi'(t)}\boldsymbol{E}X(t)X(s)dtds.$$

Let us denote L^2_X the completion of $\mathcal{D}(\boldsymbol{R})$ by above inner product. Then $I(\cdot)$ becomes an isometry form L^2_X into $L^2(\Omega)$.

Example 1. In case of Brownian motion, L^2_X is $L^2(\boldsymbol{R},dx)$ and the isometry $I(\cdot)$ is nothing but the Wiener integral.

Example 2. Let us consider a self-similar process X^α of index α (see § 3). The inner product is

$$(\varphi,\psi)=\frac{1}{2}\iint\varphi'(t)\overline{\psi'(s)}\{|t|^\alpha+|s|^\alpha-|t-s|^\alpha\}dtds$$
$$=\frac{1}{2}\alpha(\alpha-1)\iint\varphi(t)\overline{\psi(s)}\,|t-s|^{\alpha-2}\,dtds.$$

We obtain the above formula in the sense of generalized functions, the function $|t-s|^{\alpha-2}$ is accordingly considered a pseudo function (Gel'fand etc. [6]). The above inner product space is used as the space of supplementary series representation of $SL(2,\boldsymbol{R})$. Therefore this example suggests us a certain connection between self-similar process and supplementary series representation of $SL(2,\boldsymbol{R})$.

§5. Group action and unitary representation. Let us consider the case where a group G acts on the path space of a process X. We suppose

$\mathcal{G}0)$ $X^g(t;\omega)$, the transformed process, is continuous,

$\mathcal{G}1)$ X^g is subject to the same law as X, and

$\mathcal{G}2)$ $(X^g)^h(t;\omega)=X^{gh}(t;\omega)$.

The integral $I^g(\varphi;\omega)$ is introduced with respect to X^g. The inner product depends only on the covariance of the process X^g, thereby the inner product $(\ ,\ )_g$ corresponds to I^g is independent of the choice of g. We thus consider $\{I^g;\, g\in G\}$ a set of isometries from L_X^2 into $L^2(\Omega)$.

Define an operator T_g as

$$(T_g\varphi,\psi)=E[I^{g^{-1}}(\varphi)\overline{I^e(\psi)}],\qquad g\in G. \tag{8}$$

Then we have

Theorem 5. 1) *T_g is a unitary operator and*

2) *$T_gT_h=T_{gh}$, that is T_g is a unitary representation of G.*

Proof. 1) The isometric property of T_g is a direct consequence of the assumption $\mathcal{G}1)$. For any $\varphi,\psi\in\mathcal{D}$, it holds

$$\begin{aligned}(T_g\varphi,\psi)&=E\iint\varphi'(t)\overline{\psi'(s)}X^{g^{-1}}(t)X(s)dtds\\&=E\iint\varphi'(t)\overline{\psi'(s)}X(t)X^g(s)dtds=(\varphi,T_{g^{-1}}\psi).\end{aligned}$$

Thus, T_g is a unitary operator.

2) $$\begin{aligned}(T_g(T_h\varphi),\psi)=(T_h\varphi,T_{g^{-1}}\psi)&=E\iint\varphi'(s)\overline{\psi'(t)}X^{h^{-1}}(s)X^g(t)dtds\\&=E\iint\varphi'(s)\overline{\psi'(t)}(X^{h^{-1}})^{g^{-1}}(s)X(t)dtds=(T_{gh}\varphi,\omega).\end{aligned}$$ Q.E.D.

Example 1. The group action (1) defines a unitary representation of $SL(2,\boldsymbol{R})$.

Example 2. The action

$$X^{\alpha,g}(t;\omega)=|a|^{-\alpha}X^\alpha(a^2t+ab)-|a|^{-\alpha}X^\alpha(ab),\qquad g=\begin{pmatrix}a, & b\\ 0, & 1/a\end{pmatrix}\in G_u, \tag{3}$$

gives us a unitary representation of $G_u=\left\{\begin{pmatrix}a, & b\\ 0, & 1/a\end{pmatrix}\right\}\subset SL(2,\boldsymbol{R})$.

§6. Determination of the class of the representation. It is well known that Brownian motion is ergodic under time shift. The representation derived from the action (1) is accordingly irreducible. Naturally, we wish to study the class which the representation belongs to. The general theory tells us that all irreducible unitary representations of $SL(2,\boldsymbol{R})$ are classified as a unitary equivalence class by an index (Gel'fand etc. [7]).

Define infinitesimal generators as follows

$$H=\frac{d}{du}T_{\begin{pmatrix}e^{u/2}, & 0\\ 0, & e^{-u/2}\end{pmatrix}}\Big|_{u=0},\quad N_+=\frac{d}{du}T_{\begin{pmatrix}1, u\\ 0, 1\end{pmatrix}}\Big|_{u=0}\quad\text{and}\quad N_-=\frac{d}{du}T_{\begin{pmatrix}1, 0\\ u, 1\end{pmatrix}}\Big|_{u=0}.$$

In our case,

$$H=-\frac{1}{2}-t\frac{d}{dt}\quad\text{and}\quad N_+=\frac{d}{dt}.$$

The explicit form of third operator is obtained from the relation

$$\begin{pmatrix} 1, & 0 \\ k, & 1 \end{pmatrix} = -J \begin{pmatrix} 1, & -k \\ 0, & 1 \end{pmatrix} J, \qquad J = \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}.$$

For $\varphi, \psi \in \mathcal{D}(\boldsymbol{R})$, we have

$$\boldsymbol{E}\left[\int \varphi'(t) t X\left(-\frac{1}{t}\right) dt \int \psi'(s) X(s) ds\right] = \boldsymbol{E}\left[-\int \varphi'\left(-\frac{1}{t}\right) t^{-3} X(t) dt \int \psi'(s) X(s) ds\right]$$

$$= -\int_0^\infty \varphi'\left(-\frac{1}{t}\right) t^{-3} \int_0^t s\psi'(s) ds dt - \int_0^\infty \varphi'\left(-\frac{1}{t}\right) t^{-2} \int_t^\infty \psi'(s) ds dt$$

$$+ \int_{-\infty}^0 \varphi'\left(-\frac{1}{t}\right) t^{-3} \int_t^0 s\psi'(s) ds dt + \int_{-\infty}^0 \varphi'\left(-\frac{1}{t}\right) t^{-2} \int_{-\infty}^t \psi'(s) ds dt$$

$$= \int_0^\infty \phi'\left(-\frac{1}{t}\right) t^{-3} \int_0^t \psi(s) ds dt - \int_{-\infty}^0 \varphi'\left(-\frac{1}{t}\right) t^{-3} \int_t^0 \psi(s) ds dt$$

$$= \int_0^\infty \psi(s) \int_s^\infty \varphi'\left(-\frac{1}{t}\right) t^{-3} dt ds - \int_{-\infty}^0 \psi(s) \int_{-\infty}^s \varphi'\left(-\frac{1}{t}\right) t^{-3} dt ds$$

$$= \int_{-\infty}^\infty \psi(s) \frac{1}{s} \varphi\left(-\frac{1}{s}\right) ds + \int_0^\infty \psi(s) \int_{-1/s}^0 \varphi(t) dt ds + \int_{-\infty}^0 \psi(s) \int_0^{-1/s} \varphi(t) dt ds.$$

This means that

$$(T_J \varphi)(t) = \frac{1}{t} \varphi\left(-\frac{1}{t}\right) - \int_{-1/t}^0 \varphi(s) ds.$$

Finally, we obtain that

$$N_- = -J\left(\frac{d}{dt}\right) J = t^2 \frac{d}{dt} + t - \left(\frac{d}{dt}\right)^{-1},$$

where the pseudo-differential operator $(d/dt)^{-1}$ is defined as

$$\left(\frac{d}{dt}\right)^{-1} \varphi(t) = -\int_t^\infty \varphi(s) ds, \quad \text{if } t \geq 0 \text{ and } = \int_{-\infty}^t \varphi(s) ds, \quad t < 0.$$

The Laplace-Bertrami operator Δ of an irreducible representation becomes a multiple of identity operator;

$$\Delta \equiv \frac{1}{2} H^2 + \frac{1}{4}(N_+ N_- + N_- N_+) = \sigma I.$$

The scalar $\sqrt{8\sigma + 1}$ determines the equivalence class of the representation. By an easy calculation we can prove the main result.

Theorem 6. *The irreducible unitary representation derived from the projective invariance* (1) *is of discrete series of index* 2.

§ 7. **Remarks.** The operators H, N_+ and N_- satisfy the relations; $[H, N_+] = N_+$, $[H, N_-] = -N_-$ and $[N_+, N_-] = 2H$. Note that N_- is not the unique solution which satisfies the above relations with given H and N_+. From a different point of view, T. Hida etc. [2] discussed another solution $\tilde{N}_- = t^2(d/dt) + t$. This operator $\tilde{N}_-$ is unique, if we search it in local operators. They also discussed a representation which corresponds to the triple $\{H, N_+, \tilde{N}_-\}$. In the same manner in § 6, we see that their representation is of principal series of index 0.

For the invariance property (3) of a self-similar process X^α, we can apply Hida's method to extend the representation of G_u into it of $SL(2, \boldsymbol{R})$.

We obtain a triplet of local operators,

$$H=-\frac{\alpha}{2}-t\frac{d}{dt}, \quad N_{+}=-\frac{d}{dt} \quad \text{and} \quad \tilde{N}_{-}=\alpha t+t^{2}\frac{d}{dt}.$$

The corresponding unitary representation is of the supplementary series of index $|\alpha-1|$ as to be suggested in § 4.

References

[2] Hida, T., Kubo, I., Nomoto, H., and Yoshizawa, H.: On projective invariance of Brownian motion. Publ. Res. Inst. Math. Sci., 4, 595–609 (1968).
[6] Gel'fand, I. M. and Shilov, G. E.: Generalized Functions. vol. 1, Academic Press (1964).
[7] Gel'fand, I. M., Graev, M. I., and Vilenkin, N. Ya.: Generalized Functions. vol. 5, Academic Press (1966).
[8] Warner, G.: Harmonic Analysis on Semi-simple Lie Group. I, II. Springer-Verlag (1972).