9. On the Erdös-Turán Inequality on Uniform Distribution. I

By Petko D. PROINOV
Department of Mathematics, University of Plovdiv, Bulgaria

(Communicated by Shokichi IYANAGA, M. J. A., Jan. 12, 1988)

The fundamental inequality of Erdös and Turán [1] (see also the monographs [2: p. 104] and [3: p. 112]) is a quantitative version of the famous Weyl criterion for uniform distribution mod 1, and it gives an important estimate for the discrepancy of a sequence of real numbers in terms of exponential sums.

In this note we present a new general version of the Erdös-Turán inequality.

1. We recall that the *discrepancy* D_N of a finite sequence consisting of N real numbers is defined to be the oscillation on $[0,1]$ of the local discrepancy $\Delta_N(x)=A_N(x)/N-x$, where $A_N(x)$ denotes the number of those fractional parts of the terms of the sequence that are less than x. The discrepancy of a sequence of real numbers is a measure of distribution of its points mod 1. In 1948, Erdös and Turán [1] proved the following

Theorem A. *Let D_N be the discrepancy of a sequence $(x_n)_1^N$ of real numbers. Then for any positive integer m,*

$$D_N \leqq \frac{C_1}{m+1}+C_2\sum_{h=1}^{m}\frac{1}{h}\left|\frac{1}{N}\sum_{n=1}^{N}e^{2\pi i h x_n}\right|, \tag{1}$$

where C_1 and C_2 are absolute positive constants.

For a bounded function f, we denote by $\|f\|$ and $[f]$ its supremum norm and its oscillation (both on the domain of definition), respectively. For a function f of bounded variation on $[0,1]$, we denote by $\hat{f}$ its Fourier-Stieltjes transform which is given by

$$\hat{f}(h)=\int_0^1 e^{2\pi i h x}\,df(x) \qquad \text{for all integers } h.$$

In 1973, Niederreiter and Philipp [4] proved the following general version of the Erdös-Turán inequality.

Theorem B. *Let F be a nondecreasing function on $[0,1]$ with $F(0)=0$ and $F(1)=1$, and let a function G satisfy the Lipschitz condition on $[0,1]$ with constant L. Suppose also that $G(0)=0$ and $G(1)=1$. Then for any positive integer m,*

$$[F-G]<\frac{4L}{m+1}+\frac{4}{\pi}\sum_{h=1}^{m}\left(\frac{1}{h}-\frac{1}{m+1}\right)|\hat{F}(h)-\hat{G}(h)|.$$

In order to formulate a new generalization of Erdös-Turán inequality we need one more notion.

Definition. Let f be a real-valued function defined on an interval Δ, and let L be a positive number. The function f is said to satisfy the *right*

Lipschitz condition on $\varDelta$ with constant L if

$$f(x)-f(y) \leqq L(x-y) \qquad \text{for } x, y \in \varDelta \text{ with } x>y. \tag{2}$$

Analogously, f is said to satisfy the *left Lipschitz condition* on $\varDelta$ with constant L if

$$f(x)-f(y) \geqq -L(x-y) \qquad \text{for } x, y \in \varDelta \text{ with } x>y. \tag{3}$$

The function f is said to satisfy the *one-sided Lipschitz condition* on $\varDelta$ with constant L if either (2) or (3) holds.

We note that if a function f satisfies a one-sided Lipschitz condition on a closed interval $\varDelta$, then it is a function of bounded variation on $\varDelta$. Indeed, if for example f satisfies the right Lipschitz condition on $\varDelta$ with constant L, then it can be written as a difference of the nondecreasing functions $F(x)=Lx$ and $G(x)=Lx-f(x)$.

The main result of the paper is the following

Theorem 1. *Let a function f satisfy the one-sided Lipschitz condition on* $[0,1]$ *with constant L, and let $f(0)=f(1)$. Then for any positive integer m, we have*

$$[f] < \frac{4L}{m+1} + \frac{4}{\pi} \sum_{h=1}^{m} \left(\frac{1}{h} - \frac{1}{m+1} \right) |\hat{f}(h)|. \tag{4}$$

We note that Theorem A is a special case of Theorem 1 since the local discrepancy $\varDelta_N(x)$ of a sequence of real number satisfies the left Lipschitz condition on $[0,1]$ with constant $L=1$. It is easy to see that Theorem B is also a special case of Theorem 1.

(to be continued.)

References

[1] P. Erdös and P. Turán: On a problem in the theory of uniform distribution. Indag. Math., **10**, 370–378 (1948).

[2] E. Hlawka: Theorie der Gleichverteilung. Wien (1979).

[3] L. Kuipers and H. Niederreiter: Uniform distribution of sequences. Wiley (1974).

[4] H. Niederreiter and W. Philipp: Berry-Esseen bounds and a theorem of Erdös and Turán on uniform distribution mod 1. Duke Math. J., **40**, 633–649 (1973).