55. Initial Boundary Value Problem for the Equations of Ideal Magneto-Hydro-Dynamics with Perfectly Conducting Wall Condition

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1. In this paper we consider the initial boundary value problem for the equations of ideal MHD that describe the motion of an ideal plasma filling an open subset of $\boldsymbol{R}^3$, surrounded by a rigid and perfectly conducting wall. (See [1].) Our problem is to solve

$$(1)_a \quad \rho_p(\partial_t+(u\cdot\nabla))p+\rho\nabla\cdot u=0$$
$$(1)_b \quad \rho(\partial_t+(u\cdot\nabla))u+\nabla p+\mu H\times(\nabla\times H)=0$$
$$(1)_c \quad \partial_t H-\nabla\times(u\times H)=0 \qquad \text{in } [0, T]\times\Omega,$$
$$(1)_d \quad (\partial_t+(u\cdot\nabla))S=0$$
$$(1)_e \quad \nabla\cdot H=0$$
$$(2) \quad (p, u, H, S)|_{t=0}=(p_0, u_0, H_0, S_0)\equiv U_0 \qquad \text{in } \Omega,$$
$$(3) \quad u\cdot n=0,\ H\cdot n=0 \qquad \text{on } [0, T]\times\Gamma.$$

Here Ω is a bounded or unbounded domain in $\boldsymbol{R}^3$ with a smooth and compact boundary Γ, or a half space $\boldsymbol{R}^3_+$; the pressure p, the velocity $u=(u^1, u^2, u^3)$, the magnetic field $H=(H^1, H^2, H^3)$, and the entropy S are the unknown functions of t and x; the density ρ is determined by the equation of state $\rho=\rho(p, S)$; $\rho>0$ and $\rho_p=\partial\rho/\partial p>0$ for $p>0$; the magnetic permeability μ is assumed to be constant; we write $\partial_t=\partial/\partial t$, $\partial_i=\partial/\partial x_i$, $\nabla=(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$ and use the conventional notations in vector analysis; $n=n(x)=(n_1, n_2, n_3)$ denotes the unit outward normal at $x\in\Gamma$.

2. We set $U={}^t(p, u, H, S)$ and rewrite the system $(1)_{a\text{-}d}$ in the symmetric form

$$(4) \quad A_0(U)\partial_t U+\sum_{i=1}^{3} A_i(U)\partial_i U=0.$$

In order to solve the problem by iteration, we consider the linearization of (4) around an arbitrary function $U'={}^t(p', u', H', S')$ near the initial data, satisfying $u'\cdot n=0$ and $H'\cdot n=0$ on Γ. The linearized equation forms a symmetric hyperbolic system with singular boundary matrix. In fact, the boundary matrix has constant rank 2 on Γ. We define $X^m(T, \Omega)$ to be the space of functions $U(t, x)$ taking values in $\boldsymbol{R}^8$ and satisfying the following property: Let $\beta\geq 0$ be an integer and let $\Lambda_1, \cdots, \Lambda_\beta$ be an arbitrary β-tuple of smooth and bounded vector fields tangential to Γ, namely, let $\langle\Lambda_i(x), n(x)\rangle=0$ for $x\in\Gamma$, $i=1, \cdots, \beta$. Then $\partial_t^\alpha\Lambda_1\cdots\Lambda_\beta\partial_n^k U(t, x)\in L^\infty(0, T; L^2(\Omega))$ for $\alpha+\beta\leq m-2k$, $k=0, 1, \cdots, [m/2]$. Here ∂_n denotes the partial differentia-

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tion in the direction normal to Γ.

Our main results are the following two theorems.

Theorem 1. *Let Ω be a bounded domain in $\boldsymbol{R}^3$ with smooth boundary Γ. Let $m\geq 8$ be an integer. Suppose that $U_0\in H^m(\Omega)$ and that U_0 satisfies the following conditions*

$$(5)\qquad \nabla\cdot H_0=0,\ p_0>0\quad in\ \Omega,\ H_0\cdot n=0\quad on\ \Gamma,$$

$$(6)\qquad \partial_t^k u(0)\cdot n=0,\ k=0,1,\cdots,m-1,\quad on\ \Gamma.$$

Then there exists a constant $T_0>0$ such that the problem $(1)_{a\text{-}e}$ (2) (3) *has a unique solution $U\in X^m(T_0,\Omega)$.*

Theorem 2. *Let Ω be an unbounded domain in $\boldsymbol{R}^3$ with smooth and compact boundary Γ or a half space $\boldsymbol{R}^3_+$. Let $m\geq 8$ be an integer. Suppose that $U_0-{}^t(c,0)\in H^m(\Omega)$ for some constant $c>0$ and that U_0 satisfies the conditions given in Theorem* 1. *Then there exists a constant $T_1>0$ such that the problem* $(1)_{a\text{-}e}$ (2) (3) *has a unique solution U satisfying $U-{}^t(c,0)\in X^m(T_1,\Omega)$.*

Remark 1. Let U_0 satisfy $\nabla\cdot H_0=0$ in Ω and $H_0\cdot n=0$, $\partial_t^k u(0)\cdot n=0$, $k=0,1,\cdots,m-1$, on Γ. Then the solution of $(1)_{a\text{-}d}$ (2) satisfying $u\cdot n=0$ on $[0,T]\times\Gamma$ automatically satisfies $\nabla\cdot H=0$ in $[0,T]\times\Omega$, $H\cdot n=0$ on $[0,T]\times\Gamma$ and $\partial_t^k H(0)\cdot n=0$, $k=0,1,\cdots,m-1$, on Γ. This means that we may regard $\nabla\cdot H=0$ and $H\cdot n=0$ as the restrictions on the initial data U_0.

Remark 2. The characteristic boundary value problem was studied in [2]–[7]. Our approach is close to that of [6] and [7]. But some further considerations are needed for our problem.

3. Let Ω be a half space $\boldsymbol{R}^3_+=\{x|x_1>0\}$. The general case can be reduced to this case by localization and flattening of the boundary. We introduce the new unknown function $V={}^t(q-c,u,H,S)$ in place of $U={}^t(p,u,H,S)$, where $q=p+(1/2)|H|^2$ is the magnetic pressure, and rewrite the equations $(1)_{a\text{-}d}$ in the form

$$(7)\qquad \begin{pmatrix}\alpha & 0 & -\alpha H & 0\\ 0 & \rho I_3 & 0 & 0\\ -\alpha\,{}^tH & 0 & I_3+\alpha H\otimes H & 0\\ 0 & 0 & 0 & 1\end{pmatrix}\partial_t V$$

$$+\begin{pmatrix}\alpha(u\cdot\nabla) & \nabla & -\alpha H(u\cdot\nabla) & 0\\ {}^t\nabla & (u\cdot\nabla)I_3 & -(H\cdot\nabla)I_3 & 0\\ -\alpha\,{}^tH(u\cdot\nabla) & -(H\cdot\nabla)I_3 & (I_3+\alpha H\otimes H)(u\cdot\nabla) & 0\\ 0 & 0 & 0 & (u\cdot\nabla)\end{pmatrix}V$$

$$\equiv A_0(V)\partial_t V+\sum_{i=1}^3 A_i(V)\partial_i V=0.$$

Here we set $\alpha=\rho_q/\rho$ and $H\otimes H=(H^iH^j|i\rightarrow 1,2,3,\ j\downarrow 1,2,3)$. Note that $\rho=\rho(q,H)>0$, $\rho_q>0$ for $q-(1/2)|H|^2>0$. We write

$$(8)\qquad A_i(V)=\begin{pmatrix}P_i(V) & Q_i(V)\\ {}^tQ_i(V) & R_i(V)\end{pmatrix}\qquad i=0,1,2,3,$$

where $P_i(V)$, $Q_i(V)$, and $R_i(V)$ are 2×2, 2×6 and 6×6 matrices, respectively. We write also $v=(q-c,u^1)$, $w=(u^2,u^3,H^1,H^2,H^3,S)$. Hence,

$V={}^t(v, w)$. Notice that

$$P_1(V)|_{x_1=0}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad Q_1(V)|_{x_1=0}=0, \quad R_1(V)|_{x_1=0}=0,$$

if $u^1|_{x_1=0}=H^1|_{x_1=0}=0$. For a function $f(t, x)$ valued in $\boldsymbol{R}^d$, $d=6, 8$, we set

$$(9) \qquad \|f(t)\|_m^2=\sum_{k=0}^{[m/2]}\sum_{|\ell|\leq m-2k}|\partial_*^\ell \partial_1^k f(t)|_0^2, \quad \|f\|_{m,T}=\operatorname*{ess\,sup}_{t\in[0,T]}\|f(t)\|_m,$$

where $\ell=(\alpha, \beta_1, \beta_2, \beta_3)$ and $\partial_*^\ell=\partial_t^\alpha(\phi(x_1)\partial_1)^{\beta_1}\partial_2^{\beta_2}\partial_3^{\beta_3}$. The weight $\phi(x_1)$ is a smooth and positive function such that $\phi(x_1)=x_1$ for x_1 small enough and $\phi(x_1)=1$ for $x_1\geq 1$, and $|\cdot|_0$ denotes $L^2(\boldsymbol{R}_+^3)$-norm. Then $X^m(T, \boldsymbol{R}_+^3)$ consists of all functions $f(t, x)$ for which $\|f\|_{m,T}<\infty$. This is a Banach space with $\|\cdot\|_{m,T}$ taken as the norm. Now we study the linearized problem.

$$(10)_1 \qquad A_0(V')\partial_t V+\sum_{i=1}^3 A_i(V')\partial_i V=0 \ \text{ in } [0, T]\times \boldsymbol{R}_+^3,$$

$$(10)_2 \qquad V|_{t=0}=(p_0+(1/2)|H_0|^2-c, u_0, H_0, S_0)\equiv V_0 \ \text{ in } \boldsymbol{R}_+^3,$$

$$(10)_3 \qquad u^1=0 \ \text{ on } [0, T]\times\partial\boldsymbol{R}_+^3.$$

Let κ, M_{m-1}, and M_m be positive constants and let $X^m(T, \boldsymbol{R}_+^3; \kappa, M_{m-1}, M_m)$ be the set of functions V' satisfying the following conditions

$$(11) \qquad \begin{cases} V'\in X^m(T, \boldsymbol{R}_+^3),\ \partial_t^k V'(0)\in H^{m-k}(\boldsymbol{R}_+^3) & \text{for } k=0, 1, \cdots, m-1,\\ u'^1=H'^1=0 & \text{on } [0, T]\times\partial\boldsymbol{R}_+^3,\\ q'-(1/2)|H'|^2\geq\kappa & \text{for } (t, x)\in[0, T]\times\boldsymbol{R}_+^3,\\ \|V'\|_{m-1,T}\leq M_{m-1},\ \|V'\|_{m,T}\leq M_m. \end{cases}$$

Then we have

Proposition 3. *Let $m\geq 6$ and let $V'\in X^m(T, \boldsymbol{R}_+^3; \kappa, M_{m-1}, M_m)$. Then,* (i) *the null space of the boundary condition* $(10)_3$ *is the maximally non-negative subspace of the boundary matrix $-A_1(V')$ for $(t, x)\in[0, T]\times\partial\boldsymbol{R}_+^3$,* (ii) *any smooth solution of* $(10)_{1\text{-}3}$ *satisfies $H^1=0$ on $[0, T]\times\partial\boldsymbol{R}_+^3$ if $H_0^1=0$ on $\partial\boldsymbol{R}_+^3$.*

To get a counterpart of Proposition 3 for a general domain Ω, some modification is needed. In this case we add the lower order term $B(V', V)={}^t(0, 0, 0, 0, L(V', V), 0)$ to the left side of $(10)_1$, where

$$L(V', V)=\tilde{n}\{H\cdot((u'\cdot\nabla)\tilde{n})-u\cdot((H'\cdot\nabla)\tilde{n})\}$$

and $\tilde{n}=-\nabla$ dist (x, Γ). Then the assertion (ii) remains valid with modified $(10)_1$. We owe this idea to Taira Shirota.

Proposition 4. *Let $m\geq 8$ and let $V'\in X^m(T, \boldsymbol{R}_+^3; \kappa, M_{m-1}, M_m)$. Then a solution $V\in X^{m+1}(T, \boldsymbol{R}_+^3; \kappa, M_{m-1}, M_m)$ of the problem* $(10)_{1,3}$ *satisfies*

$$(12) \qquad \|V(t)\|_m\leq C(M_{m-1})\|V(0)\|_m \exp(C(M_m))t \qquad \text{for } 0\leq t\leq T.$$

Here $C(M_s)$, $s=m-1$, m, are positive constants depending only on M_s.

We now combine Propositions 3-(i) and 4 with the following arguments: (i) non-characteristic regularization (see, e.g., [5]), (ii) approximation of V' by smooth functions satisfying (11) and taking the same initial value as for V'. Then we have

Proposition 5. *Let $m\geq 8$ and let $V'\in X^m(T, \boldsymbol{R}_+^3; \kappa, M_{m-1}, M_m)$. Suppose that $V_0\in H^{m+1}(\boldsymbol{R}_+^3)$ and that V_0 satisfies conditions* (5) *and* (6). *Then the problem* $(10)_{1\text{-}3}$ *has a unique solution $V\in X^m(T, \boldsymbol{R}_+^3)$ with the estimate* (12).

By choosing the constants κ, M_{m-1}, M_m, and T suitably and by making use of Propositions 5 and 3-(ii), we can show that if $V' \in X^m(T, \boldsymbol{R}^3_+; \kappa, M_{m-1}, M_m)$, the solution V of $(10)_{1-3}$ again lies in the same set. This implies that the solution of the problem $(1)_{a-e}$ (2) (3) is constructed by iteration combined with smoothing of the initial data. Uniqueness of solution follows from the energy inequality (5.20) in [8].

Now we sketch the proof of Proposition 4. First we prove the following estimates by the standard energy method,

$$\|V(t)\|_{m,*} \leq \|V(0)\|_{m,*} + C(M_m)\int_0^t (|[v(\tau)]|_m + \|w(\tau)\|_m)d\tau, \tag{13}$$

$$\|V(t)\|_{m-1} \leq \|V(0)\|_{m-1} + C(M_{m-1})\int_0^t \|V(\tau)\|_m d\tau, \tag{14}$$

for $0 \leq t \leq T$. Here

$$\|V(t)\|^2_{m,*} = \sum_{|\ell| \leq m} |\partial^\ell_* V(t)|^2_0, \ |[v(t)]|^2_m = \sum_{k=0}^{[m/2]} \sum_{|\ell| \leq m-2k+1} |\partial^\ell_* \partial^k_1 v(t)|^2_0.$$

In deriving (13), the main terms to be estimated are the commutator parts $[\partial^\ell_*, A_1(V')]\partial_1 V$, $|\ell| \leq m$, which contain the terms such as $\partial^\nu_* Q_1(V')\partial^{\ell-\nu}_* \partial_1 w$, $\partial^\nu_* R_1(V')\partial^{\ell-\nu}_* \partial_1 w$, with $|\nu|=1$. We deal with these terms by regarding $\partial^\nu_* Q_1(V')\partial^{\ell-\nu}_* \partial_1$ and $\partial^\nu_* R_1(V')\partial^{\ell-\nu}_* \partial_1$ as the vector fields tangential to $\partial \boldsymbol{R}^3_+$. For instance, we have $\partial^\nu_* Q_1(V')\partial^{\ell-\nu}_* \partial_1 = g(V')x_1 \partial^{\ell-\nu}_* \partial_1$ where

$$g(V') = \int_0^1 \partial_1 \partial^\nu_* Q_1(V')|_{(t, \theta x_1, x_2, x_3)} d\theta,$$

because $Q_1(V')|_{x_1=0} = 0$. Similar argument was used in [5]. Second, we express $\partial_1 v$ in terms of $\partial_1 w$ and the tangential derivatives of V. Using this expression and Rauch's argument, we obtain

$$|[v(t)]|_m \leq C(M_{m-1})(\|w(t)\|_m + \|V(t)\|_{m-1} + \|V(t)\|_{m,*}). \tag{15}$$

Now we observe that w satisfies

$$R_0 \partial_t w + \sum_{i=1}^3 R_i \partial_i w = -\left({}^tQ_0 \partial_t v + \sum_{i=1}^3 {}^tQ_i \partial_i v\right).$$

In view of $R_1|_{x_1=0} = 0$, we conclude that

$$\|w(t)\|_m \leq \|w(0)\|_m + C(M_m)\int_0^t (|[v(\tau)]|_m + \|w(\tau)\|_m)d\tau \tag{16}$$

for $0 \leq t \leq T$. The estimate (12) follows from (13), (14), (15), (16), and Gronwall's inequality.

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