45. A Mathematical Theory of Randomized Computation. II

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(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1988)

Keeping up the discussion of randomized domains begun in our first note [6], we shall now establish their characterization.

Abstracting the Fatou lemma, the B. Levi theorem, and the Lebesgue dominated convergence theorem from the Lebesgue integration, we define the norm topology on a BL V as follows. We shall write $\|D\| := \{\|x\| \mid x \in D\}$ for $\forall D \subset V$:

(10) (i) The norm topology on V is *Fatou* if the norm $\|\ \|$ is a Fatou norm. A *Fatou* norm on V is a lattice norm $\|\ \|$ s.t. $D \uparrow d$ in $V^+ \Rightarrow \|d\| = \sup \|D\|$. In this case V is said to *have the Fatou property*. (ii) The norm topology on V is *Levi* if the norm $\|\ \|$ is a Levi norm. A *Levi* norm on V is a lattice norm $\|\ \|$ s.t. $D \uparrow$ in V^+ and $\sup \|D\| < \infty \Rightarrow \exists d = \sup D \in V$. In this case V is said to *have the Levi property*. (iii) The norm topology on V is *Lebesgue* if the norm is order continuous norm (or Lebesgue norm). An *order continuous* norm (or *Lebesgue* norm) on V is a lattice norm $\|\ \|$ s.t. $D \downarrow 0 \Rightarrow \|D\| \downarrow 0$. In this case V is said to *have the Lebesgue property* (or, V is *order continuous*). In (i), (ii), and (iii), if D is any countable sequence then we prefix σ- to the terms Fatou, Levi, and Lebesgue, respectively.

Now we shall identify the order topology on a randomized domain with the Scott topology: In fact, a "well defined" value x input to a randomized program, is understood to occur in the program with probability 1, thus denoting the point mass 1_x, while the undefined value $\perp$ is understood to occur in the program with probability 0, thus denoting zero probability measure 0. In higher types, a Scott continuous function corresponds to a positive order continuous operator. Moreover, it must be intuitively true that any data should be "defined with respect to the degree of definition" prior to be "measured by some metric system". So if every directed set Z has a supremum z in the Scott topology, then the metrized set $\|Z\|$ must converge to $\|z\|$ in any metric $\|\ \|$.

This observation leads us to the idea of postulating the order continuity of the norm on the BL in which a randomized domain lies. So with Vulich [5] we give the following definitions:

(11) (i) A *KB-space* is a BL whose norm topology is Levi and Lebesgue. (ii) A *KB*-space V is *algebraic* if the positive cone V^+ is an algebraic ccp

with $\perp=0$ as poset.

Then the following theorems can be deduced:

(12) (i) *If the norm topology on a BL V is Lebesgue, then it is Fatou, but the converse is not true.* (ii) *Lebesgue and Levi are independent properties of BL's.* (iii) *Every order continuous BL is order complete and order separable.* (iv) A *BL V is order continuous iff V is σ-order complete and σ-order continuous.* (v) *An AL-space is a KB-space.* (vi) *A norm separable σ-order complete BL V is order continuous.* (vii) *An order continuous BL possessing a countable basis for the order topology is norm separable.* (viii) *The positive unit hemisphere $\mathcal{H}(V)$ of an order continuous BL V is norm closed and order closed.*

6. Operator domains. Let V and W be BL's. Then:

(13) (i) A map $T: V \to W$ is an *operator* if $T(x+y)=Tx+Ty$ and $T(\lambda x)=\lambda(Tx)$ for $\forall x, y \in V$ and $\forall \lambda \in \boldsymbol{R}$. (ii) An operator $T: V \to W$ is *positive* (in symbols, $T \geq 0$) if $Tx \geq 0$ for $\forall x \in V^+$. (iii) An operator $T: V \to W$ is called *regular* if there are positive operators T_1 and T_2 s.t. $T=T_1-T_2$. $L^r(V, W)$ denotes the set of all regular operators $T: V \to W$. (iv) An operator $T: V \to W$ is called *order bounded* if T maps order intervals in V into order intervals in W. $L_b(V, W)$ denotes the set of all order bounded operators $T: V \to W$. (v) An operator $T: V \to W$ is called *σ-order continuous* if for $\forall$ sequence (x_n) in V s.t. $x_n \downarrow 0$, $\inf\{|T(x_n)| \mid n \in N\}=0$. $L_C(V, W)$ denotes the set of all σ-order continuous operators $T: V \to W$. (vi) An operator $T: V \to W$ is called *order continuous* if for $\forall D \downarrow 0$ in V^+, $\inf|T(D)| := \inf\{|T(d)| \mid d \in D\}=0$. $L_N(V, W)$ denotes the set of all order continuous operators $T: V \to W$.

By definition $L_N(V, W) \subset L_C(V, W) \subset L_b(V, W)$. If W is order complete, then $L^r(V, W)=L_b(V, W)$ and $L^r(V, W)$ is an order complete BL under the following definitions: For $\forall T, S \in L^r(V, W)$,

$(T+S)x := Tx+Sx$ and $T(\lambda x) := \lambda(Tx)$ for $\forall \lambda \in \boldsymbol{R}$,

$T \geq S$ iff $\forall x \in V[Tx \geq Sx]$,

$(T \vee S)x := \sup\{Ty+Sz \mid y \geq 0,\ z \geq 0,\ x=y+z\}$,

$(T \wedge S)x := \inf\{Ty+Sz \mid y \geq 0,\ z \geq 0,\ x=y+z\}$, and

$|T|x := \sup\{|Tz| \mid |z| \leq x\}$, for $\forall x \in V^+$.

$\|\ \|$ is the *regular* norm; $\|T\| := \inf\{\|S\| \mid \pm T \leq S \in L_b(V, W)\}$.

Moreover the following theorems can be deduced:

(14) (i) *Let V be an order continuous BL and W an order complete BL. Then $L^r(V, W)=L_N(V, W)$.* (ii) *If V and W are KB-spaces, then $L_N(V, W)$ is a KB-space and $T \in L_N^+(V, W)$ maps $\mathcal{H}(V)$ into $\mathcal{H}(W)$ iff $T \in \mathcal{H}(L_N(V, W))$.*

7. Products and exponents of KB-spaces. So we define the products and exponents of randomized domains as follows:

(15) Given KB-spaces V_1 and V_2. (i) the *product* $V_1 \times V_2$ is a normed VL in which vector and lattice operations are defined coordinatewise and the norm of $V_1 \times V_2$ is defined as the maximum of the norms on V_1 and V_2.

(ii) The *exponent* $[V \to W]$ is the KB-space $L^r(V, W)$ of all regular operators from V to W with the regular norm. The *exponent* $[\mathcal{H}(V) \to \mathcal{H}(W)]$ is $\mathcal{H}(L^r(V, W))$.

Then the following facts are easily derived:

(16) *Let V_1 and V_2 be KB-spaces. Then the product $V_1 \times V_2$ is a KB-space with order continuous projections $p_i : V_1 \times V_2 \to V_i$, $i=1, 2$. The positive unit hemisphere $\mathcal{H}(V_1 \times V_2)$ is the product $\mathcal{H}(V_1) \times \mathcal{H}(V_2)$ of the positive unit hemispheres $\mathcal{H}(V_1)$ and $\mathcal{H}(V_2)$.*

8. The characterization theorems. Now by induction on type structure randomized domains are characterized as follows:

(17) (Characterization theorem.) (i) *A randomized domain $\mathcal{R}$ is the positive unit hemisphere $\mathcal{H}$ (V) of a KB-space V.* (ii) *Mappings between randomized domains are positive operators.*

Again we note that (i) $L_N(V, W) = L^r(V, W)$ for KB-spaces, i.e., any positive operator between domains is already order continuous and norm continuous. From (12), stronger characterizations can be given of randomized domains as follows:

(18) (Strong characterization theorem.) *A randomized domain $\mathcal{R}$ can be characterized to be the positive unit hemisphere $\mathcal{H}(V)$ of one of the following BL's V:* (i) *A KB-space.* (ii) *An algebraic KB-space.* (iii) *An order separable σ-order continuous BL with the Levi property.* (iv) *A σ-order complete σ-order continuous BL with the Levi property.* (v) *A KB-space with an effectively given countable basis.* (vi) *An algebraic KB-space with the countable set of all compact elements.* (vii) *An order separable σ-order continuous BL with the Levi property and with an effectively given countable basis.* (viii) *A σ-order complete σ-order continuous BL with the Levi property and with an effectively given countable basis.* (ix) *A norm separable σ-order complete KB-space with an effectively given countable basis.*

9. An abstract integration theory. Here we consider the following stronger modification of the I-integral of Daniell [2]:

(19) Let X be a non-empty set. (i) $V \subset \boldsymbol{R}^X$ is called the *class of elementary functions* if V is an AM-space with unit under the natural vector and lattice operations and sup norm: For $\forall x, y \in V$, (a) $(x+y)(t) := x(t) + y(t)$, $(\lambda x)(t) := \lambda(x(t))$ $(\forall t \in V, \forall \lambda \in \boldsymbol{R})$, (b) $(x \vee y)(t) := \max(x(t), y(t))$, $(x \wedge y)(t) := \min(x(t), y(t))$ $(\forall t \in V)$, and (c) $\|x\| := \sup_{t \in X} |x(t)| < \infty$. (ii) I is called an *elementary integration* if I is a positive linear functional on V, i.e., (a) $I(x+y) = I(x) + I(y)$ and $I(\lambda x) = \lambda I(x)$ $(\forall x, y \in V, \forall \lambda \in \boldsymbol{R})$, (b) $x \geq 0 \Rightarrow I(x) \geq 0$, and (c) (Lebesgue property) $D \downarrow 0 \Rightarrow I(D) \downarrow 0$, where $I(D) := \{I(x) \mid x \in D\}$.

Then our integration theory implies the McShane theory [3] since the condition (c) implies his 2nd postulate by (12)-(i), and hence the Stone theory [4] by (12)-(iv). So Stone's paper can be followed without change from this point on, yielding the Bourbaki theory [1]. We define our integrability as follows:

(20) Let $\bar{R}$ be the extended reals and $\bar{V}:=\{x \in \bar{R}^X \mid D \uparrow x \subset V\}$. (i) For $\forall x \in \bar{R}^X$, the *upper integral* (or the *norm*) $N(x)$ is defined by $N(x):=\inf\{I(y) \mid y \geq |x|, y \in \bar{V}\}$. (ii) Let $\mathcal{F}:=F/\sim$, where $F:=\{x \in \bar{V} \mid N(x)<\infty\}$ and $x \sim y \Leftrightarrow N(x-y)=0$. Let $\mathcal{L}$ be the norm closure of V in $\mathcal{F}$. x in $\mathcal{L}$ is said to be *integrable*. (iii) $G(x):=N(x^+)-N(x^-)$ ($\forall x \in \mathcal{F}$), and $L:=G|\mathcal{L}$. The *general integral of* $x \in \mathcal{L}$ is given by $L(x)=N(x^+)-N(x^-)$.

Then $\mathcal{L}$ is a Banach sublattice of $\mathcal{F}$ with norm $N(x)=L(|x|)$ and the *Levi* property of $\mathcal{L}$ coincides with the integrability of the supremum of directed sets in $\mathcal{L}$ by the following generalization of the B. Levi theorem in the Lebesgue theory:

(21) *Let* $D \uparrow$ *in* $\mathcal{L}^+$. *Then* $\exists d=\sup D \in \mathcal{L}$ *iff* $\sup L(D)<\infty$.

The measurability is defined by the integrability as follows:

(22) $A \subset X$ is *measurable* iff the indicator $1_A \in \mathcal{L}$.

So the concrete KB-space $\mathcal{L}$ yields the space of Borel measures on a compact Hausdorff X in the Lebesgue theory if $V \supset C(X)$, which shows the exactitude of our characterizations in § 8.

10. The category of randomized domains and fixed point theorem. Let ***CBL*** be the category of randomized domains with positive order continuous operators. Then, from (14)–(16), we have:

(23) *The category* ***CBL*** *is cartesian closed.*

We also have the following fixed point theorem:

(24) (Fixed point theorem for randomized domains.) *Let* $\mathcal{R} \in$ ***CBL***. *Then,* (i) $\forall T \in [\mathcal{R} \to \mathcal{R}]$ *has a fixed point, and* (ii) *there exists a* Fix $\in [[\mathcal{R} \to \mathcal{R}] \to \mathcal{R}]$ *s.t. for* $\forall T \in [\mathcal{R} \to \mathcal{R}]$, Fix (T) *is the least fixed point of* T.

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