30. A Note on the Abstract Cauchy-Kowalewski Theorem

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The purpose of this note is to give a simplified proof and an extension of the nonlinear Cauchy-Kowalewski theorem established by Ovsjannikov [5], Nirenberg [3], Nishida [4] and Kano-Nishida [2] (Appendice). The formulation is generalized, and we need only the contraction mapping principle in the proofs. (See also [1] Appendix C.)

Let \( \{X_\rho; 0 \leq \rho \leq \rho_0\} \) be a Banach scale so that \( X_\rho \subset X_\rho' \) and \( |_\rho |_\rho' \) for any \( \rho_0 \leq \rho \leq \rho' \geq 0 \), where \( |_\rho | \) denotes the norm of \( X_\rho \). Consider the equation

\[
\frac{du(t)}{dt} = F(t, u(\cdot)), \quad 0 \leq t \leq T.
\]

To state the assumptions on \( F \), we introduce some notations. Let \( X_{\rho,t}(R) \) be the space of continuous functions \( f(s) \) of \( s \in [0, t] \) with values in the Banach space \( X_\rho \), which is equipped with the norm

\[
|f|_{\rho,t} = \sup_{0 \leq s \leq t} |f(s)|_\rho.
\]

We also put \( X_{\rho,t}(R) = \{f \in X_{\rho,t}; |f|_{\rho,t} \leq R\} \).

We state the assumptions on \( F \):

(F.1) There exist constants \( R > 0 \) and \( \tau_0 > 0 \) such that for any \( u \in X_{\rho,t}(R) \) \( F(t, u(\cdot)) \) is an \( X_\rho \)-valued continuous function on \( [0, \tau] \) if \( 0 \leq \rho' < \rho \leq \rho_0 - \tau_0 \).

(F.2) For \( \rho' < \rho(s) \leq \rho \leq \rho_0 - \tau_0 \) and \( 0 < \tau \leq T \), \( F \) satisfies the following inequality (3) for any \( u, v \in X_{\rho,t}(R) \):

\[
|F(t, u(\cdot)) - F(t, v(\cdot))|_\rho \
\leq \int_0^\tau C |u(s) - v(s)|_{\rho(s)} |(\rho(s) - \rho')ds,
\]

where \( C \) is a constant independent of \( t, \tau, u, v, \rho, \rho(s) \) or \( \rho' \).

(F.3) For \( 0 < \tau \leq T \) and \( \rho \leq \rho_0 - \tau_0 \), \( F(t, 0) \) is continuous in \( X_{\rho,t} \) and satisfies

\[
|F(t, 0)|_{\rho_0 - \tau_0} \leq R_0 < R.
\]

For later use we introduce two Banach spaces \( Y_{\rho,t} \) and \( Z_t \) of \( X_\rho \)-valued continuous functions, by indicating the norms (the range of \( t \) being omitted without confusion):

\[
|u|_{\rho,t} = \sup_{t \geq 0} |u(t)|_{\rho - t},
\]

\[
|u| = \sup_{0 \leq t \leq \rho - \rho} |u(t)|_{\rho \varphi(t/(\rho_0 - \rho))},
\]

where \( \varphi(t) = (1 - t)e^{-t} \). By \( Y_{\rho,t}(R) \) we denote the subset \( \{f \in Y_{\rho,t}; |f|_{\rho,t} \leq R\} \).

Clearly we have the following:

\[
\varphi(t) \text{ is monotone decreasing in } [0, 1],
\]

\[
1 - \varphi(t) > t \quad \text{for } 0 < t < 1,
\]
Theorem 1. Under the assumptions (F.1), (F.2) and (F.3) there is a constant \( \tau > \tau_0 \) such that there exists a unique solution of (1) in \( Y_{\rho_0-\tau \rho t}, X_{\rho_0-\tau \rho t} \) for \( \tau \in (0, \min\{T, \frac{\rho_0}{\tau}\}) \).

Consider an equation of extended type:

\[
(12) \quad u(t) = F(t, u(\cdot)) + \int_0^t E(t, s)G(s, u(\cdot))ds, \quad 0 \leq t \leq T.
\]

We assume that \( F \) satisfies (F.1)-(F.3) and

- (G.1) \( G(t, u(\cdot)) \) satisfies the same condition as (F.1),
- (G.2) \( \rho' < \rho(s) \leq \rho \leq \rho_0 - \tau \rho t \) there holds

\[
(13) \quad |G(t, u) - G(t, v)|_{\rho'} \leq B|u(t) - v(t)|_{\rho} + B' \int_0^t |u(s) - v(s)|_{\rho(\cdot)}|\rho(s) - \rho'|ds,
\]

where \( B \) and \( B' \) are independent of \( t, \tau, u, v, \rho, \rho(s) \) or \( \rho' \),
- (G.3) for \( 0 < \tau \leq T \) and \( \rho \leq \rho_0 - \tau \rho t \), \( G(t, 0) \) is continuous in \( X_{\rho,\tau} \) and satisfies

\[
(14) \quad |G(t, 0)|_{\rho_0-\tau \rho t} \leq R_1.
\]

We also assume the linear operator \( E(t, s) \) satisfies:

- (E) For any \( u \in X_{\rho, \tau} \), \( E(t, s)u \) is continuous on \( \Delta_{\tau} = \{(t, s) : 0 \leq s \leq t \leq T\} \) with values in \( X_{\rho, \tau} \) if \( \rho' \leq \rho - \beta(t-s) \) with some \( \beta \geq 0 \) and \( \rho \leq \rho_0 - \tau \rho t \), and there holds

\[
(15) \quad |E(t, s)u|_{\rho' - \beta(t-s)} \leq A |u|_{\rho},
\]

where the constant \( A \) does not depend on \( t, s, \rho \) or \( \rho' \).

Theorem 2. Under the assumptions (F.1)-(F.3), (G.1)-(G.3) and (E) there exists a \( \tau > \max(\tau_0, \beta \tau) \) such that there is a unique solution \( u(t) \) of (12) in \( Y_{\rho_0-\tau \rho t}, X_{\rho_0-\tau \rho t}, (R) \) for any \( \tau, 0 < \tau \leq \min(T, \frac{\rho_0}{\tau}) \).

Proof of Theorem 1. We define a mapping \( H \) from \( Z_T \cap X_{\rho, \tau} \) into \( Z_T \) by

\[
(16) \quad Hu(t) = F(t, u(\cdot)).
\]

Then we have

\[
(17) \quad |Hu(t) - Hv(t)|_{\rho} = |F(t, u(\cdot)) - F(t, v(\cdot))|_{\rho} \\
\leq \int_0^t C|u(s) - v(s)|_{\rho(\cdot)}|\rho(s) - \rho|^{-1}ds \\
\leq C \|u - v\|_{\rho} \int_0^t \varphi(\tau s/(\rho_0 - \rho(\cdot)))^{-1}(\rho(s) - \rho)^{-1}ds.
\]

We determine \( \rho(s) \) by

\[
(18) \quad \rho(s) - \rho = (\rho_0 - \rho)|\tau s/(\rho_0 - \rho)|.
\]

We can take this \( \rho(s) \), since (18) implies

\[
(19) \quad \rho_0 - \rho(s) = (\rho_0 - \rho)|1 - \varphi(\tau s/(\rho_0 - \rho))| \\
\geq (\rho_0 - \rho)(\tau s/(\rho_0 - \rho)) = \tau s.
\]

Calculating the integrand of (17), we obtain

\[
\varphi(\tau s/(\rho_0 - \rho(s))^{-1}(\rho_0 - \rho(s))^{-1} \\
\leq 2e(\rho_0 - \rho)(\rho_0 - \tau s)^{-1}e^{\tau s/(\rho_0 - \rho)}.
\]
Here we have used the equality: \(1 - e^{-x} = xe^{-x}, \ 0 < \theta < 1\), with \(x = \tilde{r}s/\rho_0 - \rho\).

Hence we have
\[
|Hu(t) - Hv(t)| e^{-\eta(t/\rho_0 - \rho)} \leq C \|u - v\| \int_0^t 2e(\rho_0 - \rho)(\rho_0 - \rho - \tilde{r}s)^{-2}ds \\
\leq 2Ce \|u - v\| (\rho_0 - \rho)^{-1}(\rho_0 - \rho - \tilde{r}t)^{-1}.
\]

This implies
\[
(20) \quad \|Hu - Hv\| \leq (2Ce/\gamma) \|u - v\|.
\]

We choose \(\gamma\) satisfying
\[
(21) \quad \gamma \geq \max (4\gamma_0/3, 8Ce).
\]

Then we define an approximating sequence \(u_n(t)\) and an associated sequence \(\gamma_n\) by
\[
(22) \quad u_0 = F(t, 0), \quad u_{n+1} = Hu_n \ (n \geq 1),
\]
\[
(23) \quad \gamma_n = \gamma(1 - 2^{-n-1}), \quad \text{i.e.} \quad \gamma - \gamma_n = r2^{-n-1},
\]

Clearly it follows (cf. (10) and (11))
\[
(24) \quad \|u_{n+1} - u_n\|_{\rho_n, \gamma_n} \leq (2Ce/\gamma_n) \|u_n - u_{n-1}\|_{\gamma_n},
\]
\[
(25) \quad \|u_{n+1} - u_n\|_{\rho_n, \gamma_n} \leq (1 - \gamma_n/\gamma)^{-1} \|u_{n+1} - u_n\|_{\gamma_n}.
\]

These imply
\[
(26) \quad \|u_{n+1} - u_n\|_{\rho_n, \gamma_n} \leq 2^{n+1}e(2Ce)^{n+1} \|u_1 - u_0\|_{\gamma_n} \leq 8/3e(4Ce/\gamma)^{n+1} \|u_1 - u_0\|_{\gamma_n},
\]

The assumptions (F.2) and (F.3) imply
\[
(27) \quad \|u_1 - u_0\|_{\gamma_1} = \|F(\cdot, u_0) - F(\cdot, 0)\|_{\gamma_1} \leq (2Ce/\gamma) \|u_0\|_{\gamma_1},
\]
\[
(28) \quad \|u_0\|_{\gamma_1} \leq \|u_0\|_{\rho_0, \gamma_0} \leq R_0.
\]

Hence we have
\[
(29) \quad \|u_{n+1} - u_n\|_{\rho_n, \gamma_1} \leq \frac{16}{9}e(4Ce/\gamma)^{n+1} R_0,
\]
\[
(30) \quad \|u_{n+1}\|_{\rho_n, \gamma_1} \leq (1 + 4e(4Ce/\gamma))R_0.
\]

If we choose \(\gamma\) satisfying (21) and
\[
(31) \quad \gamma \geq 16Ce^2R_0/(R - R_0),
\]

then we obtain
\[
(32) \quad \|u_n\|_{\rho_n, \gamma_n} \leq R, \quad n \geq 0,
\]

which shows \(u_n \in Y_{\rho_0, \gamma_1}(R)\), i.e. \(u_n(t) \in X_{\rho_0, -\gamma_1}(R)\). Thus \(Hu_n\) is well-defined.

The estimate (29) also implies that \(u_n(t)\) converges in \(Y_{\rho_0, \gamma_1}(R)\) and in \(X_{\rho_0, -\gamma_1}(R)\). The limit \(u(t)\) is a solution of (1). The uniqueness of the solution in \(Y_{\rho, \gamma}(\rho \leq \rho_0)\) is obtained from (20).

We note that the estimate of \(\gamma\) is given by
\[
(\Gamma.1) \quad \gamma = \max \{4\gamma_0/3, 8Ce, 16Ce^2R_0/(R - R_0)\}.
\]

Proof of Theorem 2. We define mappings \(K\) and \(L\) from \(Z_\gamma \cap X_{\rho, \gamma}(R)\) into \(Z_\gamma\) by
\[
(33) \quad Ku(t) = \int_0^t E(t, s)G(s, u(\cdot))ds, \quad Lu = Hu + Ku.
\]

Then from (G.1)–(G.2) and (E), we have
\[
(34) \quad |Ku(t) - Kv(t)| \leq \int_0^t AB' |u(s) - v(s)|_{\rho(t)} \{\rho(s) - (\rho + \beta(t-s))\}^{-1}ds \\
+ \int_0^t ds \int_0^s AB' |u(r) - v(r)|_{\rho(t)} \{\rho(r) - (\rho + \beta(t-s))\}^{-1}ds.
\]
We determine $\rho(s)$ by (18). From (9) we have with $r \geq 2\beta e$

$$\rho(r) - \rho \geq (\rho_0 - \rho)\left\{ \varphi(\frac{r}{\rho_0 - \rho}) - \varphi(\frac{t}{\rho_0 - \rho}) \right\}$$

$$\geq (\rho_0 - \rho)e^{-\gamma(t-r)/(\rho_0 - \rho)} \geq 2\beta(t-r),$$

(38)

Thus the same calculations as those from (19) to (20) give

$$\|Ku - Kv\| \leq (2Ae/\gamma) \|u - v\|, (B + B')\gamma, \quad (\gamma \geq \rho_0),$$

(39)

$$\|Lu - Lv\| \leq \{2(C + AB + AB'\tau_0)e/\gamma\} \|u - v\|.$$  

(40)

We choose $\tau_0$ and $\gamma$ satisfying

$$\gamma \geq \max \{4\tau_0/3, 8De, 2\beta e, \rho_0/\tau_0, 16De^2R_1/(R - R_0)\},$$

$$R_1 = R_0 + AR_1\tau_0 < R, \quad D = C + AB + AB'\tau_0.$$  

Then, we can complete the proof.

References