19. The Dimension of the Space of Relatively Invariant Hyperfunctions on Regular Prehomogeneous Vector Spaces

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(Communicated by Kôsaku YOSIDA, M. J. A., March 12, 1987)

1. Introduction. Let G_C be a connected complex algebraic group which is a linear algebraic subgroup of $GL(V_C)$ where V_C is a complex finite dimensional vector space. We suppose that there exists an irreducible non-degenerate polynomial $P(x)$ (i.e., $\det((\partial^2 P(x))/(\partial x_i \partial x_j))$ does not identically vanish) such that $P(x)$ is relatively invariant with respect to G_C and $V_C - S_C$ is the unique open orbit where $S_C := \{x \in V_C ; P(x)=0\}$. Then we have $P(g\cdot x)=\chi(g)P(x)$ for all $g \in G_C$ with a character $\chi(g)$. This means that (G_C, V_C) is a regular prehomogeneous vector space defined over the complex field C. Let V_R be a real form of V_C such that $G_R := GL(V_R) \cap G_C$ is a real form of G_C. We denote by G_R^+ the connected component of G_R containing the identity element. For a hyperfunction $f(x)$ on V_R, we say that $f(x)$ is $|\chi|^\lambda$-*invariant* ($\lambda \in C$) if it satisfies the equation $f(g\cdot x)=|\chi(g)|^\lambda f(x)$ for all $g \in G_R^+$. We denote by $\mathcal{B}^{G_R^+}(|\chi|^\lambda)$ the space of $|\chi|^\lambda$-invariant hyperfunctions on V_R. The purpose of this note is to report that, for almost all reduced regular irreducible prehomogeneous vector spaces (G_C, V_C), we can prove that the dimension of $\mathcal{B}^{G_R^+}(|\chi|^\lambda)$ coincides with l := the number of connected components of $V_R-(V_R \cap S_C)$. Moreover it is proved that they are written as a linear combination of the complex powers of $P(x)$ supported on the closures of connected components of $V_R-(V_R \cap S_R)$. It has been proved that the dimension of $\mathcal{B}^{G_R^+}(|\chi|^\lambda)$ is greater than l in a very general setting. See Muro [3], Oshima-Sekiguchi [7] and Ricci-Stein [8]. The crucial point is the upper estimate of the dimension of $\mathcal{B}^{G_R^+}(|\chi|^\lambda)$.

These results are obtained as an application of microlocal analysis. In particular, the author has already proved the same theorem for almost all real forms of regular prehomogeneous vector spaces of commutative parabolic type (defined in Muller-Rubenthaler-Schiffmann [6]) in [3]. What he wants to stress in this note is that the same method employed there works well for a wider class of regular prehomogeneous vector spaces.

2. Problem. The real locus of the open orbit $V_R-(V_R \cap S_C)$ decomposes into a finite number of connected components. Each connected component is an open G_R^+-orbit. We denote by $V_1 \cup \cdots \cup V_l$ its connected component decomposition. We define a tempered distribution $|P(x)|_i^s$ ($s \in C$ and $i=1, \cdots, l$) in the following way: When the real part of s is sufficiently large, $|P(x)|_i^s$ is defined to be a continuous function which is $|P(x)|^s$ on $x \in V_i$

and zero otherwise; it can be continued to the whole complex plane $s \in \boldsymbol{C}$ as a tempered distribution with a meromorphic parameter $s \in \boldsymbol{C}$. From the definition each $|P(x)|_i^s$ is a $|\chi|^\lambda$-invariant hyperfunction.

The problem we shall deal with here is the following:

Problem. Is any $|\chi|^\lambda$-invariant hyperfunction written as $\sum_{i=1}^{l} a_i(s) \cdot |P(x)|_i^s|_{s=\lambda}$ where $a_i(s)$ $(i=1, \cdots, l)$ are meromorphic functions defined near $s=\lambda$ such that $\sum_{i=1}^{l} a_i(s) \cdot |P(x)|_i^s$ is holomorphic with respect to s at $s=\lambda$?

For the dimension of $\mathcal{B}^{G_{\boldsymbol{R}}^+}(|\chi|^\lambda)$, the following proposition is easily proved.

Proposition. *If the claim of the above problem is valid, then the dimension of* $\mathcal{B}^{G_{\boldsymbol{R}}}(|\chi|^\lambda)$ *coincides with* $l :=$ *the number of the connected component of* $V_{\boldsymbol{R}} - (V_{\boldsymbol{R}} \cap S_{\boldsymbol{C}})$.

3. Result. The following is a conjecture by the author.

Conjecture. Let $(G_{\boldsymbol{C}}, V_{\boldsymbol{C}})$ be a regular prehomogeneous vector space defined over $\boldsymbol{C}$. For any real form $(G_{\boldsymbol{R}}, V_{\boldsymbol{R}})$ of $(G_{\boldsymbol{C}}, V_{\boldsymbol{C}})$ the answer to the above problem is always affirmative.

It seems to be difficult to justify the claim in the above problem for all the regular irreducible prehomogeneous vector spaces. However we may prove it by case-by-case calculation utilizing micro-local analysis. Indeed, the author proved that the above conjecture is actually valid for all the prehomogeneous vector spaces of commutative parabolic type (Muro [3]), which are important examples of regular irreducible prehomogeneous vector spaces. By using the same method, the author has succeeded in proving the above conjecture for almost all reduced regular irreducible prehomogeneous vector spaces.

Definition (Sato-Kimura [1]). A prehomogeneous vector space $(G_{\boldsymbol{C}}, V_{\boldsymbol{C}})$ is called *reduced* if there is no prehomogeneous vector space $(G'_{\boldsymbol{C}}, V'_{\boldsymbol{C}})$ with $\dim V'_{\boldsymbol{C}} < \dim V_{\boldsymbol{C}}$ obtained by a castling transform of $(G_{\boldsymbol{C}}, V_{\boldsymbol{C}})$.

The complete list of reduced regular irreducible prehomogeneous vector space is given in Sato-Kimura [1] § 7, pp. 144–147. They contain 29 kinds of prehomogeneous vector spaces. When $(G_{\boldsymbol{C}}, V_{\boldsymbol{C}})$ is a reduced regular irreducible prehomogeneous vector space, we can prove that the claim of the above problem is actually valid for almost all the examples except for five of them. Namely:

Theorem. *For all the real forms* $(G_{\boldsymbol{R}}, V_{\boldsymbol{R}})$ *of a reduced regular irreducible prehomogeneous vector space, the claim of the above problem is valid affirmative except for five of all the* 29 *types of reduced regular irreducible prehomogeneous vector spaces in Sato-Kimura's list* (*Sato-Kimura* [1] § 7 pp. 144–147); *those five are* (8), (10), (11), (20), *and* (21) *in Sato-Kimura's list.*

Remark. 1) As for the five exceptional cases, the author cannot apply the microlocal method since the real structure of the holonomic system to invariant hyperfunctions is complicated and the method has not been applied at present.

2) We suppose that S_C decomposes into a finite number of G_C^1-orbits with $G_C^1:=\{g \in G_C; \chi(g)=1\}$. Then any $G_R^+ \cap G_C^1$-invariant tempered distribution supported in $V_R \cap S_R$ is obtained as a finite sum of coefficients of negative degree in Laurent expansion of $|P(x)|_i^s$ at poles. See Muro [2]. We call such a distribution a singular invariant distribution.

3) The problem of determination of invariant distributions has a close relation with computation of the dimension of intertwining operators between two series of unitary representations of semi-simple Lie groups. See for example Kashiwara-Vergne [4] and Rubenthaler [5].

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