110. L^2-theory of Singular Perturbation of Hyperbolic Equations. I

A Priori Estimates with Parameter ε

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In this note and the forthcoming one, we will study asymptotic expansion of the solutions to linear Cauchy problems for a hyperbolic operator of higher order $P=(i\varepsilon)^{\nu}L+M$ with a small parameter ε in $n+1$ dimensional (t,x)-space, which reduces to an appropriate hyperbolic operator M of lower order. They have been studied mainly in the case of 2nd order in two dimensional (t,x)-space (e.g. [3], [4]) except for some references (e.g. [1], [2]).

By using pseudo-differential operators, we derive a priori L^2 estimates with ε from the separation conditions introduced by G. B. Whitham [7], [8] and completed by T. T. Wu [9]. They will give the remainder estimates of the asymptotic expansions of the solutions.

§1. Assumptions. Let S^m be the set of all C^∞ functions $a(t,x,\xi;\varepsilon)$ in $\boldsymbol{R}\times\boldsymbol{R}^n\times\boldsymbol{R}^n$ with a non-negative parameter ε in $[0,\varepsilon_0]$ such that for all j, k, α, β the derivative $\partial_\varepsilon^j\partial_\xi^\alpha\partial_t^k\partial_x^\beta a$ has the bound

(1) $\sup\{\partial_\varepsilon^j\partial_\xi^\alpha\partial_t^k\partial_x^\beta a(t,x,\xi;\varepsilon);\ 0\leqq\varepsilon\leqq\varepsilon_0,\ t>0,\ x\in\boldsymbol{R}^n\}\leqq C(1+|\xi|)^{m-|\alpha|}$

where C depends on j, k, α, β. For homogeneous symbols $a(t,x,\xi;\varepsilon)$, $b(t,x,\xi;\varepsilon)$, etc. in the sequel, the expression $a>b$ (uniformly) means that

$$\inf\{a(t,x,\xi;\varepsilon)-b(t,x,\xi;\varepsilon);\ 0\leqq\varepsilon\leqq\varepsilon_0,\ t>0,\ x\in\boldsymbol{R}^n,\ |\xi|=1\}>0,$$

and $\{a,b\}>\{c,d\}$ means that $\min\{a,b\}>\max\{c,d\}$. Let Op^m be the set of pseudo-differential operators with smooth parameters (t,ε) associated to the symbols in S^m.

Let $L(t,x,D_t,D_x;\varepsilon)=D_t^l+\sum_{j=1}^{l}L_j(t,x,D_x;\varepsilon)D_t^{l-j}$ where $L_j(t,x,D_x;\varepsilon)\in Op^j$ and let $M(t,x,D_t,D_x;\varepsilon)=\sum_{j=0}^{m}M_j(t,x,D_x;\varepsilon)D_t^{m-j}$ where $M_j(t,x,D_x;\varepsilon)\in Op^j$, and M_0 be a multiplication operator $m_0(t,x;\varepsilon)$.

We assume the following conditions (H0) and (H1).

(H0) (regular hyperbolicity of L). The operator L has its homogeneous principal symbol $l(t,x,\tau,\xi;\varepsilon)$ with the decomposition

(2) $$l(t,x,\tau,\xi;\varepsilon)=\prod_{j=1}^{l}(\tau-\varphi_j(t,x,\xi;\varepsilon))$$

where

(3) $\varphi_1(t,x,\xi;\varepsilon)<\varphi_2(t,x,\xi;\varepsilon)<\cdots<\varphi_l(t,x,\xi;\varepsilon)$ (uniformly).

(H1) (regular hyperbolicity of M). The operator M has its homogeneous principal symbol $m(t,x,\tau,\xi;\varepsilon)$ with the decomposition

$$(4) \qquad m(t, x, \tau, \xi\,;\varepsilon)=m_0(t, x\,;\varepsilon)\prod_{j=1}^{m}(\tau-\psi_j(t, x, \xi\,;\varepsilon))$$

where

$$(5) \qquad \psi_1(t, x, \xi\,;\varepsilon)<\psi_2(t, x, \xi\,;\varepsilon)<\cdots<\psi_m(t, x, \xi\,;\varepsilon) \text{ (uniformly).}$$

We study Cauchy problems to the operator $P=(i\varepsilon)^\nu L+M$. Generalizing the classification by G. B. Whitham and T. T. Wu [9], we consider the following cases.

Case 1. Let $\nu=1$. We assume

$$\text{(E)} \qquad \operatorname{Re} m_0(t, x\,;\varepsilon)>0 \text{ (uniformly)}$$

and that the characteristic roots $\{\varphi_j\}$ and $\{\psi_j\}$ separate each other such that

$$\text{(S)} \qquad \varphi_1(t, x, \xi\,;\varepsilon)<\psi_1(t, x, \xi\,;\varepsilon)<\varphi_2(t, x, \xi\,;\varepsilon)<\cdots <\psi_m(t, x, \xi\,;\varepsilon)<\varphi_{m+1}(t, x, \xi\,;\varepsilon) \text{ (uniformly).}$$

Case 2. Let $\nu=1$. We assume

$$\text{(SP)} \qquad \operatorname{Re} m_0(t, x\,;\varepsilon)=0 \text{ identically and } \operatorname{Im} m_0(t, x\,;\varepsilon)>0 \text{ (uniformly)}$$

and that $\{\varphi_j\}$ is separated weakly by $\{\psi_j\}$ such that

$$\text{(WSP)} \qquad \varphi_1(t, x, \xi\,;\varepsilon)<\{\psi_1(t, x, \xi\,;\varepsilon),\ \varphi_2(t, x, \xi\,;\varepsilon)\}<\cdots <\{\psi_{m-1}(t, x, \xi\,;\varepsilon),\ \varphi_m(t, x, \xi\,;\varepsilon)\} <\{\psi_m(t, x, \xi\,;\varepsilon),\ \varphi_{m+1}(t, x, \xi\,;\varepsilon)\} \text{ (uniformly).}$$

Alternatively, we assume

$$\text{(SN)} \qquad \operatorname{Re} m_0(t, x\,;\varepsilon)=0 \text{ identically and } \operatorname{Im} m_0(t, x\,;\varepsilon)<0 \text{ (uniformly)}$$

and

$$\text{(WSN)} \qquad \{\psi_1(t, x, \xi\,;\varepsilon),\ \varphi_1(t, x, \xi\,;\varepsilon)\}<\{\psi_2(t, x, \xi\,;\varepsilon),\ \varphi_2(t, x, \xi\,;\varepsilon)\} <\{\psi_3(t, x, \xi\,;\varepsilon),\ \varphi_3(t, x, \xi\,;\varepsilon)\}<\cdots <\{\psi_m(t, x, \xi\,;\varepsilon),\ \varphi_m(t, x, \xi\,;\varepsilon)\}<\varphi_{m+1}(t, x, \xi\,;\varepsilon) \text{ (uniformly).}$$

Case 3. Let $\nu=2$. We assume

$$\text{(P)} \qquad m_0(t, x\,;\varepsilon)>0 \text{ (uniformly),}$$

and that $\{\varphi_j\}$ is separated weakly by $\{\psi_j\}$ such that

$$\text{(WS)} \qquad \varphi_1(t, x, \xi\,;\varepsilon)<\{\psi_1(t, x, \xi\,;\varepsilon),\ \varphi_2(t, x, \xi\,;\varepsilon)\}<\cdots <\{\psi_m(t, x, \xi\,;\varepsilon),\ \varphi_{m+1}(t, x, \xi\,;\varepsilon)\}<\varphi_{m+2}(t, x, \xi\,;\varepsilon) \text{ (uniformly).}$$

§2. Results. We have the following a priori L^2 estimates for the operator P in each case mentioned above. We use the norm $\|D^k u(t)\|_p^2 =\sum_{j=0}^{k}\|D_t^j u(t, \cdot)\|_{p+k-j}^2$, where $\|\cdot\|_s^2$ denotes Sobolev norm of order s in $\boldsymbol{R}_x^n$. We omit the subscript p when $p=0$.

Theorem. *We assume* (H0) *and* (H1). *In each case there exist positive constants c, C, γ_0 such that for any $\gamma>\gamma_0$ and for $u(t)\in C^\infty([0, T]\,;$ $C_0^\infty(\boldsymbol{R}_x^n))$ the following estimates hold respectively. In Case* 1, *we have*

$$(6) \qquad C\Big\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\varepsilon^{-1}\|Pu(t)\|^2 dt+\varepsilon\|D^m u(0)\|^2+\gamma\|D^{m-1}u(0)\|^2\Big\}$$
$$\geqq c\Big\{\gamma\int_0^T e^{-2\gamma t}(\varepsilon\|D^m u(t)\|^2+\gamma\|D^{m-1}u(t)\|^2)dt + e^{-2\gamma T}(\varepsilon\|D^m u(T)\|^2+\gamma\|D^{m-1}u(T)\|^2)\Big\}.$$

In Case 2, *we have*

$$(7)\qquad C\left\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\varepsilon^{-1}\|Pu(t)\|^2dt+\varepsilon\|D^m u(0)\|^2+\|D^{m-1}u(0)\|_{1/2}^2\right\}$$
$$\geqq c\left\{\gamma\int_0^T e^{-2\gamma t}(\varepsilon\|D^m u(t)\|^2+\|D^{m-1}u(t)\|_{1/2}^2)dt\right.$$
$$\left.+e^{-2\gamma T}(\varepsilon\|D^m u(T)\|^2+\|D^{m-1}u(T)\|_{1/2}^2)\right\}.$$

In Case 3, *we have*

$$(8)\qquad C\left\{\frac{1}{\gamma}\int_0^T e^{-2\gamma t}\varepsilon^{-2}\|Pu(t)\|^2dt+\varepsilon^2\|D^{m+1}u(0)\|^2+\|D^m u(0)\|^2\right\}$$
$$\geqq c\left\{\gamma\int_0^T e^{-2\gamma t}(\varepsilon^2\|D^{m+1}u(t)\|^2+\|D^m u(t)\|^2)dt\right.$$
$$\left.+e^{-2\gamma T}(\varepsilon^2\|D^{m+1}u(T)\|^2+\|D^m u(T)\|^2)\right\}.$$

Remark. In Case 1, if $m_0(t,x;\varepsilon)$ is moreover real, the estimate is slightly improved. The proof is based on the Gårding-Leray inequality [5] extended to pseudo-differential operators:

Lemma. *We assume* (H0) *and* (H1) *for* L *and* M *with* $\nu=1$. *We assume moreover*

(P) $$m_0(t,x;\varepsilon)>0 \text{ (uniformly)}$$

and (S). *Then, there exist positive constants* c, C, γ_0 *such that for any* $\gamma>\gamma_0$ *and for* $u(t)\in C^\infty([0,T];C_0^\infty(\boldsymbol{R}_x^n))$ *such that*

$$(9)\qquad -\mathrm{Im}\int_0^T e^{-2\gamma t}(L(t,\cdot,D_t,D_x)u(t),\ M(t,\cdot,D_t,D_x)u(t))dt$$
$$\geqq c\gamma\int_0^T e^{-2\gamma t}\|D^m u(t)\|^2dt+ce^{-2\gamma T}\|D^m u(T)\|^2-C\|D^m u(0)\|^2.$$

This is proved by Euclidean algorithm for L and M (R. Sakamoto [6]). Full details will be published elsewhere.

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