107. Interaction of Two Nonlinear Waves at the Boundary

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§0. **Introduction.** In this paper, we will consider the following nonlinear mixed probrem:

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)u=f\left(\frac{\partial u}{\partial y}\right)$$
$$\text{in } \Omega_M^+=\{(t,x,y)\in \boldsymbol{R}^3\,;\,0<y,\ |t|<M\}$$
$$u|_{y=0}=0$$

where f is a smooth function which will be chosen later and M a positive number.

On the semilinear Cauchy problem, Rauch and Reed [5] have shown by a simple example that anomalous singularities arise when three characteristic hypersurfaces Σ_1, Σ_2 and Σ_3, carrying progressing waves intersect. On the other hand, Bony [2], [3] and Melrose and Ritter [4] have shown that the phenomena of interactions of singularities do not occur when two hypersurfaces Σ_1 and Σ_2 intersect.

In the case of nonlinear mixed problem, Beals and Métivier [1] have shown that when single characteristic hypersurface hits the boundary transversally, then the solution will be conormal with respect to the union of the surface and the reflected characteristic hypersurface.

We will apply the method of [5] to the nonlinear mixed problem and show by an example that anomalous singularities arise when even two hypersurfaces hit the boundary at the same time.

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§1. **Singularities of the solution to a linear problem.** In this section, we will estimate from below the singularities of the solution V of the equations

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)V=\chi_{\Gamma^+}$$
$$V|_{y=0}=0$$
$$V|_{t=0}=\left.\frac{\partial V}{\partial t}\right|_{t=0}=0.$$

Here $\Gamma^+=\{(t,x,y)\in \boldsymbol{R}^3\,;\,y>0,\ t>0,\ y<-x+\sqrt{2}\,t,\ y<x+\sqrt{2}\,t\}$ and χ_{Γ^+} is its characteristic function.

Proposition 1. Sing supp V *contains the forward light cone* C_0^+ *with vertex at origin.*

Proof. We will consider sing supp $V\cap\{t=1\}$. For general t, one can show the similar assertion.

Case 1. We show that V is not smooth across $C_0^+ \cap \{y > \sqrt{2}/2\}$. For simplicity we assume $x=0$.

Let

$$Z_1(t, x, y) = \int_0^t \iint_{R^2} E(t-\tau, x-\xi, y-\eta) \cdot \chi_\Delta(\tau, \xi, \eta) d\xi d\eta d\tau$$

where E denotes the fundamental solution of $\partial^2/\partial t^2 - \partial^2/\partial x^2 - \partial^2/\partial y^2$ and $\Delta = \{(t, x, y)\,;\, y < -x + \sqrt{2}\,t,\ y < x + \sqrt{2}\,t\}$. Hörmander's theorem implies that Z_1 is smooth across $C_0^+ \cap \{y > \sqrt{2}/2\}$.

Put $W_1 = Z_1 - V$. Then $W_1 = 0$ in $\Gamma^+ \cap \{y > \sqrt{1-x^2}, |x| < 1\}$ and W_1 is written as

$$W_1 = \int_0^t \iint_{R^2} E(t-\tau, x-\xi, y-\eta) \cdot (\chi_{\Delta \setminus \Gamma^+} + \chi_{\Gamma^-})(\tau, \xi, \eta) d\xi d\eta d\tau.$$

Here $\Gamma^- = \{(t, x, y)\,;\, t > 0,\ y < 0,\ y > -x - \sqrt{2}\,t,\ y > x - \sqrt{2}\,t\}$. Assume $(1, x, y)$ is in the interior region of C_0^+ and ε is the distance of (x, y) and $C_0^+ \cap \{t=1\}$. Then as in [5], we can show that

$$\begin{aligned} W_1 &\geqq \int_0^t \iint_{R^2} E(t-\tau, x-\xi, y-\eta) \cdot \chi_{\Delta \setminus \Gamma^+}(\tau, \xi, \eta) d\xi d\eta d\tau \\ &\geqq (\text{const})\ \varepsilon^{7/2}. \end{aligned}$$

This shows that W_1 is not smooth across $C_0^+ \cap \{y > \sqrt{2}/2\}$. Thus V is not smooth across $C_0^+ \cap \{y > \sqrt{2}/2\}$.

Case 2. We show that V is not smooth across $C_0^+ \cap \{0 < y < \sqrt{2}/2\}$. Put $\Lambda = \{(t, x, y)\,;\, y < x + \sqrt{2}\,t,\ y > -x - \sqrt{2}\,t\}$, $\Lambda^\pm = \Lambda \cap \{\pm y > 0\}$, and

$$Z_2 = \int_0^t \iint_{R^2} E(t-\tau, x-\xi, y-\eta) \cdot (\chi_{\Lambda^+} - \chi_{\Lambda^-})(\tau, \xi, \eta) d\xi d\eta d\tau.$$

As above, Z_2 is smooth near the light cone $C_0^+ \cap \{0 < y < \sqrt{2}\}$.

Let

$$\begin{aligned} W_2(t, x, y) &= Z_2(t, x, y) - V(t, x, y) \\ &= \int_0^t \iint_{R^2} E(t-\tau, x-\xi, y-\eta) \cdot (\chi_{\Lambda^+ \setminus \Gamma^+} - \chi_{\Lambda^- \setminus \Gamma^-})(\tau, \xi, \eta) d\xi d\eta d\tau. \end{aligned}$$

The integral is integrated over the intersection of $(\Lambda^+ \setminus \Gamma^+) \cup (\Lambda^- \setminus \Gamma^-)$ and $\tilde{C}^-_{(t,x,y)}$, where $\tilde{C}^-_{(t,x,y)}$ denotes the interior region of the backward light cone with vertex at (t, x, y).

It is sufficient to estimate the value of W_2 from below when $(1, x, y)$ is in $\tilde{C}_0^+$. Suppose $x < 0$, $(1, x, y) \in \tilde{C}_0^+$ and put $\varepsilon = \sqrt{1-x^2} - y$, then ε is positive. Put

$$T_1 = \{\tau \in (0, 1)\,;\, C^-_{(t,x,y)} \cap (\Lambda^- \setminus \Gamma^-) \neq \emptyset\}$$

and

$$T_2 = \{\tau \in (0, 1)\,;\, C^-_{(t,x,y)} \cap (\Lambda^- \setminus \Gamma^-) = \emptyset,\ C^-_{(t,x,y)} \cap (\Lambda^+ \setminus \Gamma^+) \neq \emptyset\}.$$

Then we have

$$\begin{aligned} W_2 &= \int_{T_1} \iint_{R^2} E(t-\tau, x-\xi, y-\eta) \cdot (\chi_{\Lambda^+ \setminus \Gamma^+} - \chi_{\Lambda^- \setminus \Gamma^-})(\tau, \xi, \eta) d\xi d\eta d\tau \\ &\quad + \int_{T_2} \iint_{R^2} E(t-\tau, x-\xi, y-\eta) \cdot (\chi_{\Lambda^+ \setminus \Gamma^+} - \chi_{\Lambda^- \setminus \Gamma^-})(\tau, \xi, \eta) d\xi d\eta d\tau \\ &= W_2^1 + W_2^2. \end{aligned}$$

$W_2^1(t, x, y)$ is positive because $E(t-\tau, x-\xi, y-\eta)$ takes constant value along the arc $\{(\xi, \eta)\,;\,(x-\xi)^2+(y-\eta)^2=\tau_0\}$ $(0<\tau_0<1-\tau)$, and for each arc the part on which the integrand is positive is longer than the other part. Thus we have

$$W_2(t, x, y) \geqq W_2^2(t, x, y) \geqq (\text{const})\, \varepsilon^{7/2}.$$

This completes the proof of Proposition 1.

Proposition 2. *For each $M>0$, there exists $c_M>0$ such that*

$$\left|\frac{\partial V}{\partial y}\right| < c_M \qquad \text{in } \Omega_M^+.$$

Proof. Let $\Omega_{M,R}^+=\Omega_M^+ \cap \{(t, x, y)\,;\, |x|<R, |y|<R\}$. It is sufficient to show that there exists $c_{M,R}>0$ such that

$$\left|\frac{\partial}{\partial y} V\right| < c_{M,R} \qquad \text{in } \Omega_{M,R}^+$$

for all $R>0$. For supp $V \cap \Omega_M^+$ is contained in $\Omega_{M,R}^+$ if R is sufficiently large. We write V as follows:

$$\begin{aligned} V(t, x, y) &= \int_0^t \iint_{R^2} E(t-\tau, x-\xi, y-\eta)\cdot \chi_{\Gamma^+}(\tau, \xi, \eta)\, d\xi\, d\eta\, d\tau \\ &\quad -\int_0^t \iint_{R^2} E(t-\tau, x-\xi, y-\eta)\cdot \chi_{\Gamma^-}(\tau, \xi, \eta)\, d\xi\, d\eta\, d\tau \\ &= V_+ - V_-. \end{aligned}$$

We will estimate $(\partial/\partial y)V_+$. It is easy to see that

$$\begin{aligned} \frac{\partial}{\partial y} V_+(t, x, y) &= \frac{1}{2\pi}\int_0^t \int_{-\infty}^{\infty} \frac{Y(t-\tau-\sqrt{(x-\xi)^2+y^2})}{\sqrt{(t-\tau)^2-(x-\xi)^2-y^2}}\, d\tau\, d\xi \\ &\quad -\frac{1}{2\pi}\int_0^t \int_0^{\infty} \frac{Y(t-\tau-\sqrt{(x-\xi)^2+(y+\xi-\sqrt{2}\,t)^2})}{\sqrt{(t-\tau)^2-(x-\xi)^2-(y+\xi-\sqrt{2}\,t)^2}}\, d\tau\, d\xi \\ &\quad -\frac{1}{2\pi}\int_0^t \int_{-\infty}^{0} \frac{Y(t-\tau-\sqrt{(x-\xi)^2+(y-\xi-\sqrt{2}\,t)^2})}{\sqrt{(t-\tau)^2-(x-\xi)^2-(y-\xi-\sqrt{2}\,t)^2}}\, d\tau\, d\xi. \end{aligned}$$

Here Y denotes the Heaviside function. If (t, x, y) is in $\Omega_{M,R}^+$, then the domain of integration of each term is contained in a compact set. Thus each term is bounded in $\Omega_{M,R}^+$. This shows that $(\partial/\partial y)V_-$ is bounded in $\Omega_{M,R}^+$ by the same way.

§2. Constructing of a nonlinear equation with interacting reflection of singularities. Using V in the last section, we shall construct the solution of the boundary value problem

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right) u = f\left(\frac{\partial u}{\partial y}\right) \qquad \text{in } \Omega_M^+$$

$$u|_{y=0}=0.$$

First of all, put $v_1(t, x, y)=(y-x+\sqrt{2}\,t)_+ -(-y-x+\sqrt{2}\,t)_+$, and $v_2(t, x, y) =(y+x+\sqrt{2}\,t)_+ -(-y+x+\sqrt{2}\,t)_+$. Here

$$X_+ = \begin{cases} X & (X \geqq 0) \\ 0 & (X<0). \end{cases}$$

Then v_1 and v_2 satisfy

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)v=0$$
$$v|_{y=0}=0.$$

Moreover for any positive δ, $u_\delta=v_1+v_2+\delta V$ satisfies the equation

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)u_\delta=\delta\chi_{\Gamma+}$$
$$u_\delta|_{y=0}=0.$$

On the other hand, $(\partial/\partial y)(v_1+v_2)\leqq 2$ if t is negative. When t is positive, $(\partial/\partial y)(v_1+v_2)=4$ in Γ^+ and $(\partial/\partial y)(v_1+v_2)\leqq 3$ outside Γ^+.

Thus we can see from Proposition 2 that $(\partial/\partial y)u_\delta\geqq 4-\delta c_M$ in $\Gamma^+\cap\Omega_M^+$ and $(\partial/\partial y)u_\delta\leqq 3+\delta c_M$ in $\Omega_M^+\backslash\Gamma^+$. We take δ so small that $\delta c_M<1/3$ and then take smooth function f_δ as

$$f_\delta(X)=\begin{cases}\delta & (X>4-1/3)\\ 0 & (X<3+1/3).\end{cases}$$

Then $f_\delta((\partial/\partial y)u_\delta)=\delta\chi_{\Gamma+}$ in Ω_M^+, and u_δ satisfies the equations

$$\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)u=f_\delta\left(\frac{\partial}{\partial y}u\right)\qquad\text{in }\Omega_M^+$$
$$u|_{y=0}.$$

Finally we consider the singularities of the solution u_δ. For negative t, sing supp u_δ is contained only four characteristic planes. But Proposition 1 assures that for positive t, sing supp u_δ contains C_0^+ besides those planes.

References

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