97. *On a Class of Partially Hypoelliptic Microdifferential Equations*

By Nobuyuki TOSE and Moto-o UCHIDA
Department of Mathematics, Faculty of Science, University of Tokyo

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§1. **Introduction.** We study a class of microdifferential equations with double characteristics which are non-hyperbolic. Explicitly, let M be a real analytic manifold with a complexification X and let P be a microdifferential operator defined in a neighborhood of $\rho_0 \in \dot{T}^*_M X$ $(=T^*_M X \setminus M)$ whose principal symbol is written as

(1) $$p=\sigma(P)=p_1+\sqrt{-1}\, q_1^{2m}\cdot p_2$$

in a neighborhood of ρ_0. Here p_1, p_2 and q_1 are homogeneous holomorphic functions of order 1, 1 and 0 respectively, which are defined in a neighborhood of ρ_0. We assume that p_1, p_2 and q_1 satisfy the following conditions (2)–(6).

(2) p_1, p_2 and q_1 are real valued on $T^*_M X$.

(3) dp_1, dp_2 and ω (the canonical 1-form of $T^*_M X$) are linearly independent at ρ_0.

(4) $\{p_1, p_2\}=0$ if $p_1=p_2=0$ where $\{\cdot,\cdot\}$ denotes Poisson bracket on $T^*_M X$.

(5) $$\{p_1, q_1\}\neq 0 \quad \text{at} \quad \rho_0.$$

(6) $$p_1(\rho_0)=p_2(\rho_0)=q_1(\rho_0)=0.$$

We give a theorem concerning the propagation of singularities of solutions to $Pu=0$ on the regular involutory submanifold

$$\Sigma=\{\rho\in \dot{T}^*_M X\,;\, p_1(\rho)=p_2(\rho)=0\}.$$

Precisely, we will show supp(u) is a union of bicharacteristic leaves of Σ for any $u\in \mathcal{C}_{M,\rho_0}$ satisfying $Pu=0$. Interesting is the fact that P is hypoelliptic in the framework of 2-microlocalization.

§2. **Preliminary.** Let M be a real analytic manifold with a complexification X and Σ be a regular involutory submanifold of $\dot{T}^*_M X$. Take a complexification Λ of Σ in T^*X. Then $\tilde{\Sigma}$ denotes the union of all bicharacteristic leaves of Λ eminated from Σ. On $T^*_{\Sigma}\tilde{\Sigma}$, M. Kashiwara constructed the sheaf $\mathcal{C}^2_{\Sigma}$ of 2-microfunctions along Σ. (See Kashiwara-Laurent [2] for details about $\mathcal{C}^2_{\Sigma}$.) We can study the properties of microfunctions on Σ precisely by $\mathcal{C}^2_{\Sigma}$. Actually, we have the following exact sequences (7) and (8).

(7) $$0\longrightarrow \mathcal{C}_{\tilde{\Sigma}}|_{\Sigma}\longrightarrow \mathcal{B}^2_{\Sigma}\longrightarrow \dot{\pi}_*(\mathcal{C}^2_{\Sigma}|_{T^*_{\Sigma}\tilde{\Sigma}\setminus\Sigma})\longrightarrow 0. \qquad (\dot{\pi}: T^*_{\Sigma}\tilde{\Sigma}\setminus\Sigma\longrightarrow\Sigma.)$$

(8) $$0\longrightarrow \mathcal{C}_M|_{\Sigma}\longrightarrow \mathcal{B}^2_{\Sigma}.$$

Here $\mathcal{C}_{\tilde{\Sigma}}$ is the sheaf of microfunctions along $\tilde{\Sigma}$ and $\mathcal{B}^2_{\Sigma}=\mathcal{C}^2_{\Sigma}|_{\Sigma}$.
Moreover there exists the canonical spectral map

(9) $$Sp^2_{\Sigma}: \pi^{-1}\mathcal{B}^2_{\Sigma}\longrightarrow \mathcal{C}^2_{\Sigma} \qquad (\pi: T^*_{\Sigma}\tilde{\Sigma}\longrightarrow\Sigma),$$

by which we set for $u \in \mathcal{C}_M|_\Sigma$, $SS^2_\Sigma(u) = \operatorname{supp}(Sp^2_\Sigma(u))$.

We regard Λ as a submanifold of $\Lambda \times \Lambda$ through the injection $T^*X \simeq T^*_{\Delta_X}(X \times X) \to T^*(X \times X)$. Here Δ_X denotes the diagonal subset of $X \times X$. Then $\tilde{\Lambda}$ expresses the union of all bicharacteristic leaves of $\Lambda \times \Lambda$ issued from Λ. On $T^*_\Lambda \tilde{\Lambda}$, Y. Laurent constructed the sheaf $\mathcal{E}^{2,\infty}_\Lambda$ of 2-microdifferential operators along Λ which act on $\mathcal{C}^2_\Sigma$. See Y. Laurent [5] for details about $\mathcal{E}^{2,\infty}_\Lambda$.

§ 3. Statement of the main theorem. We follow the notation prepared in § 1. Then we give

Theorem 1. *Let u be a microfunction defined in a neighborhood of ρ_0 satisfying $Pu=0$. Then* supp (u) *is contained in* $\sum = \{\rho \in T^*_M X\,;\, p_1(\rho) = p_2(\rho) = 0\}$ *in a neighborhood of ρ_0. Moreover $SS^2_\Sigma(u)$ is contained in the zerosection Σ of $T^*_\Sigma \tilde{\Sigma}$ and thus* supp (u) *is a union of bicharacteristic leaves of Σ.*

§ 4. Proof of the main theorem. By finding a suitable real quantized contact transformation, we may assume from the beginning that P is defined in a neighborhood of $\rho_0 = (0, \sqrt{-1}\,dx_n)$ and has a form:

(10) $$P = D_1 + \sqrt{-1}\,\theta(x, D) x_1^{2m} D_2 + (\text{lower order}).$$

Here we take a coordinate of $T^*_M X$ [resp. T^*X] as $(x, \sqrt{-1}\,\xi \cdot dx)$ [resp. $(z, \zeta \cdot dz)$] with $x = (x_1, \cdots, x_n)$, $\xi = (\xi_1, \cdots, \xi_n) \in \boldsymbol{R}^n$ [resp. $z = (z_1, \cdots, z_n)$, $\zeta = (\zeta_1, \cdots, \zeta_n) \in \boldsymbol{C}^n$] and $\theta(x, D)$ is real elliptic of order 0 at ρ_0. Then we find in this case

(11) $$\Sigma = \{(x, \sqrt{-1}\,\xi \cdot dx) \in T^*_M X\,;\, \xi_1 = \xi_2 = 0\}.$$

The 1st claim of the theorem is assured by M. Sato *et al.* [7]. Moreover the equation $Pu=0$ is hypoelliptic outside Σ. This fact easily follows from the fact that P_0 is microhyperbolic in the direction dx_1 or $-dx_1$ at any point on $\{\rho \in \dot{T}^*_M X \backslash \Sigma\,;\, x_1(\rho) = \xi_1(\rho) = 0\}$.

We employ the theory of 2-microlocalization along Σ to see the phenomenon on Σ of solutions to (10). We take a coordinate of $T^*_\Sigma \tilde{\Sigma}$ as $(x, \sqrt{-1}\,\xi'' \cdot dx''\,;\, \sqrt{-1}\,(x_1^* \cdot dx_1 + x_2^* \cdot dx_2))$ with $\xi'' = (\xi_3, \cdots, \xi_n)$, $x'' = (x_3, \cdots, x_n) \in \boldsymbol{R}^{n-2}$ and $(x_1^*, x_2^*) \in \boldsymbol{R}^2$. We see easily that

(12) $$ch^2_\Lambda(P) \cap (T^*_\Sigma \tilde{\Sigma} \backslash \Sigma) = \{x_1 = x_1^* = 0\}$$

where $ch^2_\Lambda(P)$ denotes the microcharacteristic variety of P along Λ. (See § 3.1.1 of Y. Laurent [5] for the definitions of $ch^2_\Lambda(\cdot)$.)

We can find a real suitable quantized bicanonical transformation through which the equation $Pu=0$ is transformed into $P_0 u=0$ defined in a neighborhood of $\tau_0 = (0, \sqrt{-1}\,dx_n\,;\, \sqrt{-1}\,dx_2) \in T^*_\Sigma \tilde{\Sigma}$ with

(13) $$\sigma_\Lambda(P_0) = z_1^* + \sqrt{-1}\,z_1^{2m} z_2^*.$$

Here we take a coordinate of $T^*_\Lambda \tilde{\Lambda}$ as $(z, \zeta'' \cdot dz''\,;\, z_1^* dz_1 + z_2^* dz_2)$ with $\zeta'' = (\zeta_3, \cdots, \zeta_n) \in \boldsymbol{C}^{n-2}$ and $(z_1^*, z_2^*) \in \boldsymbol{C}^2$. This fact can be shown by essentially the same way as M. Sato *et al.* [7]. Using Theorem 4.4 of N. Tose [9] (see also [11]), we can find an invertible $Q \in \mathcal{E}^{2,\infty}_\Lambda$ satisfying

(14) $$P_0 Q = Q(D_1 + \sqrt{-1}\,x_1^{2m} D_2).$$

Thus it suffices to study 2-microlocally at τ_0 the equation

(15) $$P_1 u = (D_1 + \sqrt{-1}\,x_1^{2m} D_2) u = 0.$$

For any $u \in \mathcal{C}^2_{\Sigma,\tau_0}$ satisfying (15), we see by (12)

(16) $$\mathrm{supp}\,(u) \subset \{x_1 = x_1^* = 0\}.$$

Moreover $-dx_1$ is 2-microhyperbolic for P along Σ at τ_0. This means

(17) $$(-H)(-dx_1) \in C_{T^*_{\Sigma}\tilde{\Sigma}}(C_{\tilde{\Sigma}}(ch(P_1))).$$

Here $C_*(\cdot)$ denotes the normal cone defined in [3] and H is Hamiltonian isomorphism $H: T^*T^*_{\Sigma}\tilde{\Sigma} \simeq T_{T^*_{\Sigma}\tilde{\Sigma}}T^*\tilde{\Sigma}$ induced from $H: T^*T^*\tilde{\Sigma} \simeq TT^*\tilde{\Sigma}$. Then the conditions (16) and (17) implies that $u=0$ at τ_0. This fact can be shown using Theorem 5.2.1 of Kashiwara-Schapira [4]. (See also §2 of [10] for the definition of 2-microhyperbolicity.)

After all, we find that $SS^2_{\Sigma}(u) \subset \Sigma$ for $u \in \mathcal{C}_M$ satisfying (10). Thus we have shown that $u \in \mathcal{C}_{\tilde{\Sigma}}$ and that u has unique continuation property along bicharacteristic leaves of Σ. This implies the assertion of the theorem.

(*q.e.d.*)

Remark 2. The assertion of propagation of singularities itself can be verified in a more direct way using microlocal version of Holmgren's theorem. Consider the equation (10) and take any microfunction solution u to (10). Then by Bony-Schapira [12] or Theorem 9.2.1 of Kashiwara-Schapira [3], we see easily that $\mathrm{supp}\,(u) \backslash \{x_1=0\}$ is a union of bicharacteristic leaves of $\Sigma \backslash \{x_1=0\}$. Moreover dx_1 is non microcharacteristic for P along $\tilde{\Sigma}$ at any point of $\{x_1=\xi_1=0\}$, which implies immediately the assertion of propagation of singularities above. We emphasize here in this note that P is hypoelliptic in the frame work of 2-microlocalization.

Remark 3. Consider the case that a microdifferential operator P defined in a neighborhood of $\rho_0 \in \dot{T}^*_M X$ has the principal symbol of the form: $\sigma(P) = p_1 + \sqrt{-1}\,q_1^{2k+1}p_2$ where p_1, p_2 and q_1 satisfy the same conditions as in §1. We can also show

$$\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X/\mathcal{E}_X P, \mathcal{C}^2_{\Sigma})_{\tau} = 0$$

for any $\tau \in T^*_{\Sigma}\tilde{\Sigma} \backslash \Sigma$ satisfying $(ad^r \sigma_A(P))^{2k+1}\overline{\sigma_A(P)(\tau)} > 0$. Here we denote the relative Hamiltonian vector field H^r_f by ad^r_f. (See [5] for H^r_f.) The proof can be given in the same way as in this section using the 2nd semihyperbolicity of P and the exact sequence

(18) $$0 \longrightarrow \mathcal{C}^2_{\Gamma|\tilde{\Sigma}} \longrightarrow \mathcal{C}^2_{\Sigma} \longrightarrow \bigoplus_{\pm} \mathcal{C}^2_{\Sigma_{\pm}|\tilde{\Sigma}} \longrightarrow 0.$$

Refer to M. Uchida [13] for the notion of 2nd semihyperbolicity and the sheaves appearing in (18).

§5. Remark. A few words about the existence of microfunction solutions to (10). Theorem 1 shows that we have only to consider solutions with holomorphic parameters (z_1, z_2). We set $N = \{x \in M;\, x_1 = 0\}$ and take a complexification Y of N in X. We take a coordinate of $T^*_N Y$ as $(\tilde{x}, \sqrt{-1}\tilde{\xi}\cdot d\tilde{x})$ with $\tilde{x} = (x_2, \cdots, x_n)$ and $\tilde{\xi} = (\xi_2, \cdots, \xi_n) \in \boldsymbol{R}^2$. Since Y is non-microcharacteristic for P along $\tilde{\Sigma}$, we have a natural isomorphism

(20) $$\mathcal{H}om_{\mathcal{E}_X}(\mathcal{E}_X/\mathcal{E}_X P_0, \mathcal{C}_{\tilde{\Sigma}})|_{Y \times_X \tilde{\Sigma}} \xrightarrow{\sim} \mathcal{C}_{\tilde{\Sigma}_0}.$$

Here Σ_0 is a regular involutory submanifold of $T^*_N Y$:

$$\Sigma_0 = \{(\tilde{x}, \sqrt{-1}\tilde{\xi}\cdot d\tilde{x}) \in T^*_N Y;\, \xi_2 = 0\}.$$

The above isomorphism can be deduced from a result of P. Schapira [8].

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