80. A Generalization of Itô's Lemma

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1. Introduction. In the traditional definition of K. Itô's stochastic integral of a process \( \varphi \) with respect to Brownian motion \( B \) it is essential that \( \varphi \) be non-anticipatory [8]. However, there are some works in which one has tried to avoid this condition, s. e.g. [1, 4, 9]. Finally, the white noise analysis, advocated by T. Hida (e.g. [2, 3]), has provided a framework, in which stochastic integrals can be naturally defined without posing such measurability conditions, as has been shown in a recent paper by H.-H. Kuo and A. Russek [7].

Let \((\mathcal{S}'(\mathbb{R}), \mathcal{B}, \mu)\) be white noise, i.e. \( \mathcal{B} \) is the \( \sigma \)-algebra over \( \mathcal{S}'(\mathbb{R}) \) generated by the cylinder sets and \( \mu \) is the Gaussian measure on \( \mathcal{B} \) with characteristic functional

\begin{equation}
\exp \left( -\frac{1}{2} \| \xi \|^2 \right) = \int_{\mathcal{S}'(\mathbb{R})} \exp(i \langle x, \xi \rangle) d\mu(x)
\end{equation}

for \( \xi \in \mathcal{S}(\mathbb{R}) \), \( \| \cdot \|_2 \) denoting the norm of \( L^2(\mathbb{R}, dt) \) and \( \langle \cdot, \cdot \rangle \) the canonical duality. By \((L^p)\), \( p > 0 \), we denote the Banach space \( L^p(\mathcal{S}'(\mathbb{R}), \mathcal{B}, d\mu) \). Note that

\begin{equation}
B(t; x) := \langle x, 1_{(0,t]} \rangle, \quad x \in \mathcal{S}'(\mathbb{R})
\end{equation}

(although not pointwise defined) is a well-defined random variable in \((L^p)\), \( p \geq 1 \), and a Brownian motion (under \( d\mu \)).

In [2, 3] Hida introduced the space \((\mathcal{L}^\omega)^*\) of testfunctionals of white noise and its dual \((\mathcal{L}^\omega)^-\) of generalized functionals. Furthermore he defined the operators \( \partial_\omega \), \( t \in \mathbb{R} \), which are partial derivatives \( \partial/\partial x(t) \) for white noise testfunctionals, cf. also [5, 6]. Since \( \partial_\omega \) is densely defined on \((\mathcal{L}^\omega)^+\) there is its adjoint \( \partial_\omega^* \) acting on \((\mathcal{L}^\omega)^-\). Note that we have the Gel'fand triple

\begin{equation}
(\mathcal{L}^\omega)^- \subset (\mathcal{L}^\omega) \subset (\mathcal{L}^\omega)^+
\end{equation}

so that \( \partial_\omega^* \) acts by restriction on \((\mathcal{L}^\omega)\).

The following was shown in the paper [7] of Kuo and Russek: assume that \( \varphi \) is a map from \( \mathbb{R} \) into \((\mathcal{L}^\omega)\), non-anticipatory (i.e. for each \( t \in \mathbb{R} \), \( \varphi(t) \) is measurable w.r.t. \( \sigma(\mathcal{B}(s; \cdot), 0 \leq s \leq t) \)) and

\begin{equation}
\int_a^b E(|\varphi(t)|^2) dt
\end{equation}

is finite, then

\begin{equation}
\int_a^b \partial_\omega^* \varphi(t) dt
\end{equation}

exists in \((\mathcal{L}^\omega)\) and equals Itô's stochastic integral of \( \varphi \) w.r.t. Brownian motion. Of course, this generalizes to higher-dimensional Brownian
motions, using independent copies of white noise.

The important thing about (1.5) is that this expression is defined independently of any measurability conditions posed on $\varphi(t)$ (except for $\mathcal{B}$-measurability). Only (1.4) has to be supplemented with the condition that
\[ \int_{[a, b]} E(\partial_s \varphi(t) \partial_s \varphi(s)) ds dt \] be finite (cf. [7] for the action of $\partial_s$ on $(L^p)$). Hence (1.5) is an extension of the traditional definition for quite general processes in $(L^p)$.

For the theory and applications of stochastic integrals Itô’s lemma [8] plays a key role. Naturally the question arises, what this lemma becomes, if the non-anticipatory condition on $\varphi$ is dropped and stochastic integrals are understood as in (1.5). The answer is given in the next section.

2. A generalization of Itô’s Lemma. Let $\varphi_i, \psi_i, i=1, 2, \ldots, n$ be real, strongly continuous processes in $(L^p)$, such that the stochastic integrals
\[ X_i(t) := x_i + \int_0^t \partial_s^* \varphi_i(s) ds + \int_0^t \psi_i(s) ds \]
exist in $(L^p)$ for all $t \in R_+$. Here $x_i \in R$, $i=1, 2, \ldots, n$.

The discussion after (1.5) implies, that we have to set
\[ \int_0^t \eta(s) dX_i(s) = \int_0^t \partial_s^* \eta(s) \varphi_i(s) ds + \int_0^t \eta(s) \psi_i(s) ds \]
for the stochastic integral of another process $\eta$ w.r.t. $dX_i$. (2.2) guarantees that this stochastic integral coincides with the conventional one, if all processes involved are non-anticipatory w.r.t. Brownian motion.

For the following let $\eta: R_+ \rightarrow (L^p)$ be continuous and assume that $\eta(t)$ has a piecewise continuous integral kernel (cf. [2, 3, 6]). Furthermore let $(\mathcal{A})$ be a partition of $(0, t)$ into intervals $(t_i, t_{i+1})$ of length $\Delta$ and put
\[ Y_i(a, b) = \int_a^b \partial_s^* \varphi_i(s) ds, i=1, 2, \ldots, n \]

Lemma.

a) (2.4) \[ \lim_{\Delta \downarrow 0} \sum_{i} \eta(t_i) Y_i(t_i, t_{i+1}) = \int_0^t [\partial_s^* \eta(s) \varphi_i(s) + (\partial_s \eta(s)) \varphi_i(s)] ds \]

where $\partial_s$ is the derivative defined in [7]

b) (2.5) \[ \lim_{\Delta \downarrow 0} \sum_{i} \eta(t_i) Y_i(t_i, t_{i+1}) Y_j(t_i, t_{i+1}) = \int_0^t \eta(s) \varphi_i(s) \varphi_j(s) ds \]

c) (2.6) \[ \lim_{\Delta \downarrow 0} \sum_{i} \eta(t_i) Y_i(t_i, t_{i+1}) Y_j(t_i, t_{i+1}) Y_k(t_i, t_{i+1}) = 0 \]

(all limits are taken in the topology of $(L^p)$).

The proof of this lemma is performed by straightforward computations of the $S$-transforms [3, 5, 6] of products of random variables and standard estimations of the resulting Lebesgue integrals.

Now let $X(t)$ denote the $R^n$-valued random variable with components
X_i(t), i=1, 2, \ldots, n, and consider its composite F \circ X(t) = F(X(t)) with a real C(R^n) function F. Furthermore we have to assume that F and its partial derivatives D^\alpha F, |\alpha|=1, 2, 3, composed with X(t) belong to (L^2) for all t.

Under the preceding conditions it is a matter of applying Taylor's theorem and the lemma to establish the following theorem:

**Theorem.**

\[
F(X(t)) - F(X(s)) = \int_s^t \sum_{i=1}^n D^i F(X(u))dX_i(u)
\]

(2.7)

\[
+ \int_s^t \sum_{i,j=1}^n D^{ij} F(X(u))[1/2\varphi_i(u)\varphi_j(u) + \varphi_i(u)\varphi_j(u)\partial_u X_j(u)]d\lambda(u)
\]

Note that formula (2.7) reduces to the usual Itô-lemma, in case that \(\varphi_i\) and \(\psi_i\) are non-anticipatory for all \(i \in \{1, \ldots, n\}\), since then also all the \(X_i\) are non-anticipating and \(\partial_u X_i(u) = 0\) by a theorem in [7].

**References**